Notes on extremal and tame valued fields

Anscombe, Sylvy and Kuhlmann, F

Anscombe, Sylvy and Kuhlmann, F (2016) Notes on extremal and tame valued fields. Journal of Symbolic Logic, 81 (3). p. 887. ISSN 0022-4812

NOTES ON EXTREMAL AND TAME VALUED FIELDS

SYLVY ANSCOMBE AND FRANZ-VIKTOR KUHLMANN

Abstract. We extend the characterization of extremal valued fields given in [2] to the missing case of valued fields of mixed characteristic with perfect residue field. This leads to a complete characterization of the tame valued fields that are extremal. The key to the proof is a model theoretic result about tame valued fields in mixed characteristic. Further, we prove that in an extremal valued field of finite p-degree, the images of all additive polynomials have the optimal approximation property. This fact can be used to improve the axiom system that is suggested in [8] for the elementary theory of Laurent series fields over finite fields. Finally we give examples that demonstrate the problems we are facing when we try to characterize the extremal valued fields with imperfect residue fields. To this end, we describe several ways of constructing extremal valued fields; in particular, we show that in every $\aleph_1$ saturated valued field the valuation is a composition of extremal valuations of rank 1.

1. Introduction

A valued field (K, v) with valuation ring $\mathcal{O}$ and value group vK is called **extremal** if for every multi-variable polynomial $f(X_1, \ldots, X_n)$ over K the set

$$\{v(f(a_1, \ldots, a_n)) \mid a_1, \ldots, a_n \in \mathcal{O}\} \subseteq vK \cup \{\infty\}$$

has a maximal element. For the history of this notion, see [2]. In that paper, extremal fields were characterised in several special cases, but some cases remained open. In the present paper we answer the question stated after Theorem 1.2 of [2] to the positive, thereby removing the condition of equal characteristic from the theorem. The most comprehensive version of the theorem now reads:

Theorem 1.1. *Let (K, v) be a nontrivially valued field. If (K, v) is extremal, then it is algebraically complete and*

(i) *vK is a $\mathbb{Z}$-group, or*
(ii) *vK is divisible and Kv is large.*

Conversely, if (K, v) is algebraically complete and

(i) *$vK \simeq \mathbb{Z}$, or vK is a $\mathbb{Z}$-group and $\mathrm{char}\, Kv = 0$, or*
(ii) *vK is divisible and Kv is large and perfect,*

then (K, v) is extremal.

Note that a valued field (K, v) is called **algebraically complete** if every finite algebraic extension (L, v) satisfies

$$[L : K] \;=\; (vL : vK)[Lv : Kv]\,, \tag{1}$$

where Lv, Kv denote the respective residue fields. Every algebraically complete valued field (K, v) is **henselian**, i.e., v admits a unique extension to its algebraic

Date: February 9, 2016.

closure $\tilde{K}$ (which we will again denote by v). Also, every algebraically complete valued field (K, v) is **algebraically maximal**, that is, does not admit proper algebraic immediate extensions (L, v) (**immediate** means that $vL = vK$ and $Lv = Kv$). For later use let us mention that a valued field is called **maximal** if it does not admit proper immediate extensions at all.

Further, (K, v) is a **tame field** if it is henselian, perfect, and $\tilde{K}$ is equal to the ramification field of the extension $(\tilde{K}|K, v)$. All tame fields are algebraically complete (cf. [13, Lemma 3.1]).

A field K is **large** if every smooth curve over K which has a K-rational point, has infinitely many such points. For more information about large fields, see [15], [10] and [2].

In [2] it was proved that an algebraically complete valued field (K, v) with divisible value group and large perfect residue field is extremal if $\operatorname{char} K = \operatorname{char} Kv$ (the **equal characteristic** case). To this end, we used the Ax–Kochen–Ershov Principle

$$vK \equiv vL \;\wedge\; Kv \equiv Lv \quad\Longrightarrow\quad (K, v) \equiv (L, v) \tag{2}$$

which holds for all tame valued fields of equal characteristic (see [13, Theorem 1.4]). We were not able to cover the **mixed characteristic** case $\operatorname{char} K \neq \operatorname{char} Kv$ because the principle was not known for this case. In fact, we will show below (Theorem 1.5) that it is false. However, we can do with lesser tools that *are* known. After all, at least the corresponding Ax–Kochen–Ershov Principle for elementary extensions has been proved in [13]:

Theorem 1.2. *If $(L|K, v)$ is an extension of tame fields such that $vK \prec vL$ and $Kv \prec Lv$, then $(K, v) \prec (L, v)$.*

This theorem enables us to prove:

Theorem 1.3. *Take a nontrivially valued tame field (K, v) and two ordered abelian groups Γ and Δ such that $\Gamma \prec vK$ and $\Gamma \prec \Delta$. Then there exist two tame fields (K', v) and (L, v) with $vK' = \Gamma$, $vL = \Delta$, $Kv = K'v = Lv$, $(K', v) \prec (K, v)$ and $(K', v) \prec (L, v)$. In particular, $(K, v) \equiv (L, v)$.*

If vK is nontrivial and divisible and Δ is any nontrivial divisible ordered abelian group, then we can take $\Gamma = \mathbb{Q}$ to obtain that $\Gamma \prec vK$ and $\Gamma \prec \Delta$ since the elementary class of nontrivial divisible ordered abelian groups is model complete. Thus, Theorem 1.3 yields the following result:

Corollary 1.4. *If (K, v) is a nontrivially valued tame field with divisible value group and Δ is any nontrivial divisible ordered abelian group, then there is a tame field $(L, v) \equiv (K, v)$ with $vL = \Delta$ and $Lv = Kv$.*

It is easy to see that (2) cannot hold in this generality in the mixed characteristic case. One can construct two algebraic extensions (L, v) and (L', v') of $(\mathbb{Q}, v_p)$, where v_p is the p-adic valuation on $\mathbb{Q}$, both having residue field $\mathbb{F}_p$, such that:
1) L does not contain $\sqrt{p}$ and vL is the p-divisible hull of $(v_p p)\mathbb{Z}$,
2) L' contains $\sqrt{p}$ and $v'L'$ is the p-divisible hull of $(v_p \sqrt{p})\mathbb{Z} = \frac{1}{2}(v_p p)\mathbb{Z}$.
Then $vL \simeq v'L'$ and hence $vL \equiv v'L'$, but $(L, v) \not\equiv (L', v')$.

One could hope, however, that this problem vanishes when one strengthens the conditions by asking that vL and $v'L'$ are equivalent over $v_p\mathbb{Q}$ (and Lv and $L'v'$ are equivalent over $\mathbb{Q}v_p$). But the problem remains:

Theorem 1.5. *Take any prime p. Then the there exist valued field extensions*

$$(\mathbb{Q}, v_p) \subset (L_0, v) \subset (L_1, v) \subset (L_2, v) \quad \text{and} \quad (\mathbb{Q}, v_p) \subset (F_0, v) \subset (F_1, v) \subset (F_2, v)$$

such that the following assertions hold:

a) The fields (L_0, v) and (F_0, v) are extensions of degree $p(p-1)$ of the henselization of $\mathbb{Q}$ under the p-adic valuation and extremal with $L_0 v = \mathbb{F}_p = F_0 v$ and $vL_0 = vF_0 = \frac{1}{p(p-1)}(v_p p)\mathbb{Z}$, but $(L_0, v) \not\equiv (F_0, v)$.

b) The fields (L_1, v) and (F_1, v) are algebraic over $\mathbb{Q}$ and tame with $L_1 v = F_1 v = \mathbb{F}_p$ and $vL_1 = vF_1$ equal to the p-divisible hull of $\frac{1}{p-1}(v_p p)\mathbb{Z}$, but $(L_1, v) \not\equiv (F_1, v)$.

c) The fields (L_2, v) and (F_2, v) are tame and extremal, with perfect residue fields $L_2 v = F_2 v$ and $vL_2 = vF_2 = \mathbb{Q}$, but $(L_2, v) \not\equiv (F_2, v)$.

Corollary 1.6. *The Ax–Kochen–Ershov Principle (2) fails for extremal fields with value group isomorphic to $\mathbb{Z}$ in mixed characteristic. It also fails for tame extremal fields with value group isomorphic to $\mathbb{Q}$ and perfect residue field in mixed characteristic.*

Open problem: Can the situation be improved by adding the Macintyre power predicates to the language?

Note that $(L, v) \equiv (L', v')$ if and only if they are equivalent over $(\mathbb{Q}, v_p)$, and this in turn holds if and only if we have the equivalence

$$(L, v)_\delta \equiv (L', v')_\delta \quad \text{over} \quad (\mathbb{Q}, v_p)_\delta$$

of their amc structures of level δ, for all $\delta \in (v_p p)\mathbb{Z}$ (see [7, Corollary 2.4]). But this fact is of little use for the proof of Corollary 1.4 since it is by no means clear how to construct an extension of $(\mathbb{Q}, v_p)$ whose amc structures of level δ are equivalent to those of (K, v).

The improvement in Theorem 1.1 yields a corresponding improvement of Proposition 5.3 from [2]. Note that when we speak of a composition $v = w \circ \overline{w}$ of valuations, we do not mean a composition as functions, but in fact refer to the composition of their associated places. That is, if Q and $\bar{Q}$ are the places associated with w and $\bar{w}$, then their composition (with the obvious additional rules for ∞) is the place associated with v.

Proposition 1.7. *Take a valued field (K, v) with perfect residue field. Assume that v is the composition of two nontrivial valuations: $v = w \circ \overline{w}$. Then (K, v) is extremal with divisible value group if and only if the same holds for (K, w) and $(Kw, \overline{w})$.*

We may say that a property P of valuations is **compatible with composition** if $P(v) \Leftrightarrow P(w) \wedge P(\bar{w})$ for each composition $v = w \circ \bar{w}$. Examples of such properties are "henselian", "maximal", "algebraically complete", "divisible value group". The latter two will be used in the proof of the proposition, given in Section 2. The proposition in fact yields that also the property "extremal with divisible value group and perfect residue field" is compatible with composition (since if $(Kw, \bar{w})$ has this property, then in particular it is perfect).

It should be noted that the condition on the value groups cannot be dropped without a suitable replacement, even when all residue fields have characteristic 0. Indeed, if the value group of (K, w) is a $\mathbb{Z}$-group and $\overline{w}$ is nontrivial, then the value group of (K, v) is neither divisible nor a $\mathbb{Z}$-group and (K, v) cannot be extremal.

Let us state two
Open problems:

1) If $v = w \circ \bar{w}$ with w and $\bar{w}$ extremal and w having divisible value group, does it follow that v is extremal?

2) We know that if $v = w \circ \bar{w}$ is extremal, then so is $\bar{w}$ (see Lemma 4.1 below). But we do not know whether it follows that also w is extremal.

Tame fields of positive residue characteristic $p > 0$ are algebraically complete, and by [13, Theorem 3.2], they have p-divisible value groups which consequently are not $\mathbb{Z}$-groups. On the other hand, by the same theorem all algebraically complete valued fields with divisible value group and perfect residue field are tame fields. Therefore, in the case of positive residue characteristic and value groups that are not $\mathbb{Z}$-groups, the above Theorem 1.1 is in fact talking about tame fields:

Theorem 1.8. *A tame field of positive residue characteristic is extremal if and only if its value group is divisible and its residue field is large.*

Again, we see that we know almost everything about tame fields (with the exception of quantifier elimination in the case of equal characteristic), but almost nothing about imperfect valued fields. As shown in [2], there are some algebraically complete valued fields with value group a $\mathbb{Z}$-group and a finite residue field that are extremal, and others that are not. In particular, the Laurent series field $\mathbb{F}_q((t))$ over a finite field $\mathbb{F}_q$ with q elements is extremal.

It is a longstanding open question whether $\mathbb{F}_q((t))$ has a decidable elementary theory. However, in recent years progress has been made on the existential theory. Denef and Schoutens showed in [4] that if Resolution of Singularity holds in positive characteristic in all dimensions (which is a longstanding open problem), then the existential theory of $(\mathbb{F}_q((t)), t)$ — i.e., the field together with the constant t — is decidable. More recently, Anscombe and Fehm showed in [1] that the existential theory of $\mathbb{F}_q((t))$ is decidable, under no assumptions.

Since the question for the full elementary theory has remained open, it is important to search for a complete recursive axiomatization. Such an axiomatization was suggested in [8], using the elementary property that the images of additive polynomials have the optimal approximation property (see Section 3 for the definition of this notion). For the case of $\mathbb{F}_q((t))$, this was proved in [3]. At first sight, extremality seems to imply the optimal approximation property for the images of additive polynomials. But the latter uses inputs from the whole field while the former restricts to inputs from the valuation ring. However, we will prove in Section 3:

Theorem 1.9. *If (K, v) is an extremal field of characteristic $p > 0$ with $[K : K^p] < \infty$, then the images of all additive polynomials have the optimal approximation property.*

Open problem:

Does the assertion of this theorem fail in the case of $[K : K^p] = \infty$?

Since the elementary property of extremality is more comprehensive and easier to formulate than the optimal approximation property, it is therefore a good idea to replace the latter by the former in the proposed axiom system for $\mathbb{F}_q((t))$. We also note that every extremal field is algebraically complete by Theorem 1.1. So we ask:

Open problem: Is the following axiom system for the elementary theory of $\mathbb{F}_q((t))$ complete?
1) (K, v) is an extremal valued field of positive characteristic,
2) vK is a $\mathbb{Z}$-group,
3) $Kv = \mathbb{F}_q$.

In order to obtain the assertion of Theorem 1.9 in the case of algebraically complete perfect fields of positive characteristic (which are exactly the tame fields of positive characteristic), one does not need the assumption that the field be extremal. Indeed, S. Durhan recently proved in [5]:

Theorem 1.10. *If (K, v) is a tame field of positive characteristic, then the images of all additive polynomials have the optimal approximation property.*

There are tame fields of positive characteristic that are not extremal, e.g. the power series field $\mathbb{F}_p((\Gamma))$ with Γ the p-divisible hull of $\mathbb{Z}$ (see Theorem 1.8). Therefore, the previous theorem yields:

Corollary 1.11. *There are perfect non-extremal fields of positive characteristic in which the images of all additive polynomials have the optimal approximation property.*

Open problem:
Is there an imperfect non-extremal field of characteristic $p > 0$ in which the images of all additive polynomials have the optimal approximation property?

Finally, let us point out that we still do not have a complete characterization of extremal fields:

Open problem: Take a valued field (K, v) of positive residue characteristic. Assume that vK is a $\mathbb{Z}$-group, or that vK is divisible and Kv is an imperfect large field. Under which additional assumptions do we obtain that (K, v) is extremal?

Additional assumptions are indeed needed, as we will show in Section 4:

Proposition 1.12. *a) There are algebraically complete valued fields (K, v) of positive characteristic and value group a $\mathbb{Z}$-group that are extremal, and others that are not.*

b) There are algebraically complete valued fields (K, v) of mixed characteristic with value group a $\mathbb{Z}$-group that are extremal, and others that are not.

c) There are algebraically complete nontrivially valued fields (K, v) of positive characteristic with divisible value group and imperfect large residue field that are extremal, and others that are not.

d) There are algebraically complete valued fields (K, v) of mixed characteristic with divisible value group and imperfect large residue field that are extremal, and others that are not.

None of the non-extremal fields that we construct for the proof of parts a)–d) of this proposition is maximal. This leads us to the following

Conjecture: Every maximal field with value group a $\mathbb{Z}$-group, or divisible value group and large residue field, is extremal.

The following theorem, also proved in Section 4, provides a compelling way of constructing maximal extremal fields and is used in the proof of parts c) and d) of the previous theorem.

Theorem 1.13. *Let (K, v) be any $\aleph_1$-saturated valued field. Assume that Γ and Δ are convex subgroups of vK such that $\Delta \subsetneq \Gamma$ and Γ/Δ is archimedean. Let u (respectively w) be the coarsening of v corresponding to Δ (resp. Γ). Denote by $\bar{u}$ the valuation induced on Kw by u. Then $(Kw, \bar{u})$ is maximal, extremal and large, and its value group is isomorphic either to $\mathbb{Z}$ or to $\mathbb{R}$. In the latter case, also $Ku = (Kw)\bar{u}$ is large.*

Remark 1.14. Pairs (Γ, Δ) of convex subgroups satisfying the conditions of this theorem are abundant and can easily be constructed. Indeed, for any $\gamma \in vK$ we can take Γ to be the smallest convex subgroup of vK containing γ (the intersection of all convex subgroups of vK containing γ), and Δ to be the largest convex subgroup of vK not containing γ (the union of all convex subgroups of vK not containing γ). Then Δ is the largest proper convex subgroup of Γ and therefore, Γ/Δ is archimedean.

From Theorem 1.13 we can derive an interesting observation about infinite compositions of henselian valuations. Note that every valuation can be viewed as a possibly infinite composition of rank 1 valuations, i.e., valuations with archimedean ordered value groups. It is well known that $v = w \circ \bar{w}$ is henselian if and only if both w and $\bar{w}$ are. However, in Section 4 we will derive the following result:

Corollary 1.15. *There exist non-large (and therefore non-henselian) valued fields (K, v) with the following property: if $v = w_1 \circ w_2 \circ w_3$ with w_2 of rank 1, then w_2 is henselian and both Kw_1 and $(Kw_1)w_2$ are large.*

The part about henselianity also follows from an actually stronger result, stating the existence of a non-henselian valued field (K, v) with the following property: if $v = w_1 \circ w_2$ with nontrivial w_1, then w_2 is henselian; see [12, Proposition 4]. The latter again implies that Kw_1 is large, but we do not know how to show that the field constructed in the cited paper is not large.

Acknowlegements.
Several of the ideas contained in this paper were conceived at a 2 hour seminar talk the second author gave to the logic group at the University of Wroclaw in Poland. The audience was arguably the most lively and inspiring the author has ever witnessed. He would like to thank this group for the great hospitality.

The authors would like to thank the referee for his careful reading of the manuscript and for several very useful suggestions that inspired them to come up with Theorem 1.13 and with a new version of Theorem 1.5.

The second author would like to thank Koushik Pal for proofreading an earlier version of the paper, and Anna Blaszczok for very helpful corrections and comments on a later version.

During this research, the first author was funded by EPSRC grant EP/K020692/1, and the second author was partially supported by a Canadian NSERC grant and a sabbatical grant from the University of Saskatchewan.

2. Proof of Theorems 1.1, 1.3 and 1.5, and Proposition 1.7

As a preparation, we need a few basic facts about tame fields. For the following lemma, see [13, Lemma 3.7]:

Lemma 2.1. *Take a tame field (L,v). If K is a relatively algebraically closed subfield of L such that $Lv|Kv$ is algebraic, then (K,v) is a tame field, vL/vK is torsion free, and $Lv = Kv$.*

We derive:

Corollary 2.2. *Take a tame field (K,v) and an ordered abelian group $\Gamma \subset vK$ such that vK/Γ is torsion free. Then there exists a tame subfield (K',v) of (K,v) with $vK' = \Gamma$ and $K'v = Kv$.*

Proof. Denote the prime field of K by K_0 and note that $k_0 := K_0v$ is the prime field of Kv. Take a maximal system γ_i, $i \in I$, of elements in Γ rationally independent over vK_0. Choose elements $x_i \in K$ such that $vx_i = \gamma_i$, $i \in I$. Further, take a transcendence basis t_j, $j \in J$, of Kv over its prime field, and elements $y_j \in K$ such that $y_jv = t_j$ for all $j \in J$. For $K_1 := K_0(x_i, y_j \mid i \in I, j \in J)$ we obtain from [13, Lemma 2.2] that $vK_1 = vK \oplus \bigoplus_{i\in I} \gamma_i\mathbb{Z}$ and $K_1v = k_0(t_j \mid j \in J)$, so that Γ/vK_1 is a torsion group and $Kv|K_1v$ is algebraic.

Now we take K' to be the relative algebraic closure of K_1 in K. Then by Lemma 2.1, (K',v) is a tame field with vK/vK' torsion free and $K'v = Kv$. Since $\Gamma \subseteq vK$ and Γ/vK_2 is a torsion group, we have that $\Gamma \subseteq vK'$. Since vK/Γ is torsion free, we also have that $vK' \subseteq \Gamma$, so that $vK' = \Gamma$. □

Lemma 2.3. *Take a tame field (K,v) and an ordered abelian group Δ containing vK such that Δ is p-divisible, where p is the characteristic exponent of Kv. Then there exists a tame extension field (L,v) of (K,v) with $vL = \Delta$ and $Lv = Kv$.*

Proof. By Theorem 2.14 of [9] there is an extension (K_1,v) of (K,v) such that $vK_1 = \Delta$ and $K_1v = Kv$. We take (L,v) to be a maximal immediate algebraic extension of (K_1,v); then (L,v) is algebraically maximal. Since $vL = vK_1 = \Delta$ is p-divisible, and $Lv = K_1v = Kv$ is perfect by [13, Theorem 3.2] applied to (K,v), it follows from the same theorem that (L,v) is a tame field. □

Now we can give the

Proof of Theorem 1.3: Since $\Gamma \prec vK$ by assumption, we have that vK/Γ is torsion free. Hence by Corollary 2.2 we find a tame subfield (K',v) of (K,v) with $vK' = \Gamma$ and $K'v = Kv$. Again since $\Gamma \prec vK$, it follows from Theorem 1.2 that $(K',v) \prec (K,v)$.

Since (K',v) is a tame field, we know that $\Gamma = vK'$ is p-divisible. As $\Gamma \prec \Delta$, the same holds for Δ. Hence by Lemma 2.3 we can find a tame extension field (L,v) of (K',v) with $vL = \Delta$ and $Lv = K'v$. Since $vK' = \Gamma \prec \Delta = vL$, it follows again from Theorem 1.2 that $(K',v) \prec (L,v)$. □

Theorem 1.3 is the key to the

Proof of Theorem 1.1: In view of Theorems 1.2 and 4.1 of [2], we only have to show that if (K,v) is algebraically complete with divisible value group and large perfect residue field, then (K,v) is extremal. Note that (K,v) is then a tame field, being algebraically complete with perfect residue field and p-divisible value group.

Every trivially valued field is extremal, so we may assume that (K,v) is non-trivially valued. We apply Corollary 1.4 with $\Delta = \mathbb{R}$ to obtain a tame field

$(L,v) \equiv (K,v)$ with value group $vL = \mathbb{R}$. By the proof of Theorem 1.2 in [2], this field is extremal. Since extremality is an elementary property, also (K,v) is extremal. $\square$

We turn to the
Proof of Theorem 1.5: We extend the p-adic valuation v_p of $\mathbb{Q}$ to some valuation v on the algebraic closure of $\mathbb{Q}$. Adjoining a primitive p-th root of unity ζ_p to $\mathbb{Q}$ and passing to the henselization $K := \mathbb{Q}(\zeta_p)^h = \mathbb{Q}^h(\zeta_p)$, we obtain that $vK = \frac{1}{p-1}(v_p p)\mathbb{Z}$ and $Kv = \mathbb{Q}v_p = \mathbb{F}_p$.

By general ramification theory, the Galois extension $\mathbb{F}_{p^p}|\mathbb{F}_p$ can be lifted to a Galois extension of degree p of K. Since K contains the p-th roots of unity, Kummer theory shows that this extension is generated by an arbitrary p-th root of some element $b \in K$.

Now we take L_0 (respectively, F_0) to be the Galois extension of K generated by a p-th root of bp (resp., of p). Then $\frac{1}{p(p-1)}(v_p p)\mathbb{Z} \subseteq vL_0$ and $\frac{1}{p(p-1)}(v_p p)\mathbb{Z} \subseteq vF_0$, and since

$$[L_0 : K] \;=\; [F_0 : K] \;=\; p \;=\; \left(\frac{1}{p(p-1)}(v_p p)\mathbb{Z} \,:\, vK\right) ,$$

the fundamental inequality $n \geq ef$ shows that

$$vF_0 \;=\; vL_0 \;=\; \frac{1}{p(p-1)}(v_p p)\mathbb{Z} \quad \text{and} \quad L_0 v \;=\; F_0 v \;=\; \mathbb{F}_p \,.$$

Since both (L_0,v) and (F_0,v) are henselian fields of characteristic 0 with value group isomorphic to $\mathbb{Z}$, they are algebraically complete. Hence by [2, Theorem 4.1], both fields are extremal.

Next, in order to construct (L_1,v) and (F_1,v), we choose algebraic extensions (L_1',v) of (L_0,v) and (F_1',v) of (F_0,v) such that $vL_1' = vF_1'$ is the p-divisible hull of $vL_0 = vF_0$ and hence of $\frac{1}{p-1}(v_p p)\mathbb{Z}$, and $L_1'v = L_0v = \mathbb{F}_p = F_0v = F_1'v$; this is possible by [9, Theorem 2.14].

Now we take (L_1,v) (resp., (F_1,v)) to be a maximal immediate algebraic extension of (L_1',v) (resp., (F_1',v)). Then by [13, Theorem 3.2], (L_1,v) and (F_1,v) are tame fields. Their value groups and residue fields are as in the assertion of part b) of Theorem 1.5.

Finally, in order to construct (L_2,v) and (F_2,v), we choose an arbitrary nontrivially valued henselian and perfect field (k,w) of characteristic p such that $kw = \mathbb{F}_p$. (For example, we could take the power series field $\mathbb{F}_p((t^{\mathbb{Q}}))$ for k and the t-adic valuation for w; but also the much smaller relative algebraic closure of $\mathbb{F}_p(t)$ in $\mathbb{F}_p((t^{\mathbb{Q}}))$ works.) Using [9, Theorem 2.14] again, we construct extensions (L_2',v) of (L_1,v) and (F_2',v) of (F_1,v) such that $vL_2' = vF_2' = \mathbb{Q}$ and $L_2'v = F_2'v = k$. As before, we take (L_2,v) (resp., (F_2,v)) to be a maximal immediate algebraic extension of (L_2',v) (resp., (F_2',v)). Then again by [13, Theorem 3.2], (L_2,v) and (F_2,v) are tame fields. Since their residue field k admits a nontrivial henselian valuation, it is a large field. Hence by Theorem 1.1, (L_2,v) and (F_2,v) are also extremal.

It remains to show that (L_i,v) and (F_i,v) are not elementarily equivalent, for $i = 1,2,3$. Assume the contrary. Then L_0 and F_0 or L_1 and F_1 would be isomorphic over $\mathbb{Q}$, as all of them are algebraic over $\mathbb{Q}$. Likewise, if (L_2,v) and (F_2,v) are elementarily equivalent then we obtain an isomorphism of the algebraic parts of L_2 and F_2 over $\mathbb{Q}$. In all three cases, this yields an embedding of F_0 in L_2 and hence the existence of all p-th roots of p in L_2. But L_2 also contains a p-th root of bp, hence a p-th root of b as well. This however contradicts the fact that by construction, $(L_2v)w$ does not contain $\mathbb{F}_{p^p}$. □

We conclude this section with the

Proof of Proposition 1.7: In both directions we assume that Kv is perfect.

First we assume that (K,v) is extremal and vK is divisible. By the compatibility of "divisible value group" with composition, both wK and $\bar{w}(Kw)$ are divisible. Theorem 1.1 shows that (K,v) is algebraically complete and that $Kv = (Kw)\bar{w}$ is large. By the compatibility of "algebraically complete" with composition, both (K,w) and $(Kw,\bar{w})$ are algebraically complete. The latter has a large perfect residue field, hence by Theorem 1.1, it is extremal. As in addition its value group is divisible and its residue field is perfect, it is itself perfect. Since Kw carries the nontrivial henselian valuation $\bar{w}$, it is large (see e.g. [10, Proposition 16]). Therefore, also (K,w) has a large perfect residue field, and again it follows from Theorem 1.1 that it is extremal.

For the converse, we assume that both (K,w) and $(Kw,\bar{w})$ are extremal with divisible value group. By Theorem 1.1, both are algebraically complete, with large residue fields. By compatibility it follows that (K,v) is algebraically complete with divisible value group. We know that $Kv = (Kw)\bar{w}$ is large, and it is also perfect by assumption. Now Theorem 1.1 shows that (K,v) is extremal.

3. Additive polynomials over extremal fields

We start by introducing a more precise notion of extremality. Take a valued field (K,v), a subset S of K, and a polynomial f in n variables over K. Then we say that (K,v) **is S-extremal with respect to** f if the set $vf(S^n) \subseteq vK \cup \{\infty\}$ has a maximum. We say that (K,v) **is S-extremal** if it is S-extremal with respect to every polynomial in any finite number of variables. With this notation, (K,v) being extremal means that it is $\mathcal{O}$-extremal, where $\mathcal{O}$ denotes the valuation ring of (K,v).

A subset A of a valued field (K,v) has the **optimal approximation property** if for every $z \in K$ there is some $y \in A$ such that $v(z-y) = \max\{v(z-x) \mid x \in A\}$. A polynomial $h \in K[X_1,\ldots,X_n]$ is called a **p-polynomial** if it is of the form $f+c$, where $f \in K[X_1,\ldots,X_n]$ is an additive polynomial and $c \in K$. The proof of the following observation is straightforward:

Lemma 3.1. *The images of all additive polynomials over (K,v) have the optimal approximation property if and only if K is K-extremal with respect to all p-polynomials over K.*

We will work with ultrametric balls

$$B_\alpha(a) := \{b \in K \mid v(a-b) \geq \alpha\},$$

where $\alpha \in vK$ and $a \in K$. Observe that $\mathcal{O} = B_0(0)$. We note:

Proposition 3.2. *Take $\alpha, \beta \in vK$ and $a, b \in K$. Then (K,v) is $B_\alpha(a)$-extremal if and only if it is $B_\beta(b)$-extremal. In particular, (K,v) is $B_\alpha(a)$-extremal if and only if it is extremal.*

Proof. It suffices to prove that "$B_\alpha(a)$-extremal" implies "$B_\beta(b)$-extremal". Take a polynomial f in n variables. If $c \in K$ is such that $vc = \beta - \alpha$, then the function $y \mapsto c(y-a)+b$ establishes a bijection from $B_\alpha(a)$ onto $B_\beta(b)$. We set $g(y_1,\ldots,y_n) := f(c(y_1-a)+b,\ldots,c(y_n-a)+b)$. It follows that $f(B_\beta(b)^n) = g(B_\alpha(a)^n)$, whence $vf(B_\beta(b)^n) = vg(B_\alpha(a))^n$. Hence if (K,v) is $B_\alpha(a)$-extremal with respect to g, then it is $B_\beta(b)$-extremal with respect to f. This yields the assertions of the proposition. □

A valued field (K,v) of characteristic $p>0$ is called **inseparably defectless** if every finite purely inseparable extension $(L|K,v)$ satisfies equation (1) (note that the extension of v from K to L is unique). This holds if and only if every finite subextension of $(K|K^p,v)$ satisfies equation (1).

If (K,v) is inseparably defectless with $[K:K^p]<\infty$, then for every $\nu\geq 1$, the extension $(K|K^{p^\nu},v)$ has a **valuation basis**, that is, a basis of elements $b_1,\ldots,b_\ell$ that are **valuation independent** over K^{p^ν}, i.e.,

$$v(c_1b_1+\ldots+c_\ell b_\ell) \;=\; \min_{1\leq i\leq \ell} vc_ib_i$$

for all $c_1,\ldots,c_\ell \in K^{p^\nu}$.

Note that every algebraically complete valued field is in particular inseparably defectless. By Theorem 1.1, every extremal field is algebraically complete and hence inseparably defectless.

Proposition 3.3. *Take an inseparably defectless valued field (K,v) with $[K:K^p]<\infty$ and an additive polynomial f in n variables over K. Then for some integer $\nu\geq 0$ there are additive polynomials $g_1,\ldots,g_m\in K[X]$ in one variable such that*

a) $f(K^n) = g_1(K)+\ldots+g_m(K)$,
b) all polynomials g_i have the same degree p^ν,
c) the leading coefficients $b_1,\ldots,b_m$ of $g_1,\ldots,g_m$ are valuation independent over K^{p^ν}.

Proof. The proof can be taken over almost literally from Lemma 4 of [3]. One only has to replace the elements $1,t,\ldots,t^{\delta_i-1}$ from that proof by an arbitrary basis of $K|K^{\delta_i}$. □

The following theorem is a reformulation of Theorem 1.9 of the Introduction.

Theorem 3.4. *Assume that (K,v) is an extremal field of characteristic $p>0$ with $[K:K^p]<\infty$. Then it is K-extremal w.r.t. all p-polynomials and therefore, the images of all additive polynomials have the optimal approximation property.*

Proof. Take a p-polynomial h in n variables over K, and write it as $h=f+c$ with f an additive polynomial in n variables over K and $c\in K$. We choose additive polynomials $g_1,\ldots,g_m\in K[X]$ in one variable satisfying assertions a), b), c) of Proposition 3.3. Then $h(K^n) = g_1(K)+\ldots+g_m(K)+c$.

We write $g_i = b_iX^{p^\nu} + c_{i,\nu-1}X^{p^{\nu-1}} + \ldots + c_{i,0}X$ for $1\leq i\leq m$. Then we choose $\alpha\in vK$ such that

$$\alpha \;<\; \min\{0,\, vc-vb_i\,,\, vc_{i,k}-vb_i \mid 1\leq i\leq m\,,\, 0\leq k<\nu\}\,.$$

Because $\alpha<0$, it then follows that for each a with $va\leq\alpha$,

$$vb_i + p^\nu va \;\leq\; vb_i + p^\nu\alpha \;\leq\; vb_i+\alpha \;<\; vc$$

and for $0 \leq k < \nu$,

$$vb_i + p^\nu va \;\leq\; vb_i + va + p^k va \;\leq\; vb_i + \alpha + p^k va \;<\; vc_{i,k} + p^k va\,.$$

It then follows that

$$vg_i(a) \;=\; vb_i + p^\nu va \;\leq\; vb_i + p^\nu \alpha \;<\; vc\,. \tag{3}$$

On the other hand, if $va' \geq \alpha$, then $vb_i + p^\nu va' \;\geq\; vb_i + p^\nu \alpha$ and $vc_{i,k} + p^k va' \;\geq\; vc_{i,k} + p^k \alpha \;>\; vb_i + p^\nu \alpha$ for $0 \leq k < \nu$. This yields that

$$vg_i(a') \;\geq\; vb_i + p^\nu \alpha\,. \tag{4}$$

Now take any $(a_1', \ldots, a_m') \in B_\alpha(0)^n$ and $(a_1, \ldots, a_m) \in K^n \setminus B_\alpha(0)^n$. So we have:

$$\min\{va_1, \ldots, va_m\} \;<\; \alpha \;\leq\; \min\{va_1', \ldots, va_m'\}\,.$$

Since $b_1, \ldots, b_m$ are valuation independent over K^{p^ν}, we then obtain from (3) and (4) that

$$\begin{aligned} vh(a_1, \ldots, a_m) \;&=\; \min_{1\leq i\leq m} vb_i + p^\nu va_i \\ &<\; \min_{1\leq i\leq m} vb_i + p^\nu \alpha \;\leq\; vh(a_1', \ldots, a_m')\,. \end{aligned}$$

This proves that

$$vh(B_\alpha(0)^n) \;>\; vh(K^n \setminus B_\alpha(0)^n)\,.$$

Since (K, v) is extremal by assumption, Proposition 3.2 shows that $vh(B_\alpha(0)^n)$ has a maximal element, and the same is consequently true for $vh(K^n)$. This shows that (K, v) is K-extremal w.r.t. h, from which the first assertion follows. The second assertion follows by Lemma 3.1. □

4. More constructions of extremal fields, and proof of Theorem 1.13

It follows from [2, Theorem 4.1] that the Laurent series fields $(\mathbb{F}_p((t)), v_t)$ and the p-adic fields $(\mathbb{Q}_p, v_p)$ are extremal. The former have equal characteristic, the latter mixed characteristic. All of them have $\mathbb{Z}$ as their value group, which is a $\mathbb{Z}$-group.

In [8] a valued field extension (L, v) of $(\mathbb{F}_p((t)), v_t)$ is presented in which not all images of additive polynomials have the optimal approximation property. In [2] it is shown that (L, v) is not extremal, although it is algebraically complete and its value group vL is a $\mathbb{Z}$-group (of rank 2). It is also shown that for the nontrivial coarsening w of v corresponding to the convex subgroup $(v_t t)\mathbb{Z}$ of vL, also (L, w) is not extremal. As a coarsening of an algebraically complete valuation, it is also algebraically complete. Its value group $wL = vL/(v_t t)\mathbb{Z}$ is divisible and its residue field $Lw = \mathbb{F}_p((t))$ is large, but not perfect. Note that (L, v) and (L, w) are of equal characteristic.

In order to prove the remaining existence statements of Proposition 1.12 concerning non-extremal fields in mixed characteristic, we consider compositions of valuations. Unfortunately, contrary to the assertion that the proof of Lemma 5.2 of [2] is easy (and thus left to the reader), we are unable to prove it in the cases that are not covered by Proposition 1.7. (However, we also do not know of any counterexample.) In fact, a slightly different version can easily be proved: *If (K, v) is $\mathcal{O}_v$-extremal, then also (K, w) is $\mathcal{O}_v$-extremal.* We do not know whether the latter impies that (K,w) is $\mathcal{O}_w$-extremal. Proposition 3.2 is of no help here because $\mathcal{O}_v$ is in general not a ball of the form $B_\alpha(a)$ in (K,w).

It appears, though, that we actually had in mind the following result, which is indeed easy to prove:

Lemma 4.1. *If (K,v) is extremal and $v = w\circ\overline{w}$, then $(Kw,\overline{w})$ is extremal.*

Proof. Assume that (K,v) is extremal with $v = w\circ\overline{w}$; note that for any $a,b\in\mathcal{O}_w$, $\overline{w}(aw) > \overline{w}(bw)$ implies $va > vb$.

Assume further that $g \in Kw[X_1,\dots,X_n]$. Then choose $f \in \mathcal{O}_w[X_1,\dots,X_n]$ such that $fw = g$. By assumption, there are $b_1,\dots,b_n\in\mathcal{O}_v$ such that

$$vf(b_1,\dots,b_n) \;=\; \max\{vf(a_1,\dots,a_n)\mid a_1,\dots,a_n\in\mathcal{O}_v\}\,.$$

Since $b_1,\dots,b_n\in\mathcal{O}_v\subseteq\mathcal{O}_w$ we have that

$$f(b_1,\dots,b_n)w \;=\; fw(b_1w,\dots,b_nw) \;=\; g(b_1w,\dots,b_nw)\,.$$

We claim that

$$\overline{w}g(b_1w,\dots,b_nw) \;=\; \max\{\overline{w}g(\overline{a}_1,\dots,\overline{a}_n)\mid \overline{a}_1,\dots,\overline{a}_n\in\mathcal{O}_{\overline{w}}\}\,.$$

Indeed, if there were $\overline{a}_1,\dots,\overline{a}_n\in\mathcal{O}_{\overline{w}}$ with $\overline{w}g(\overline{a}_1,\dots,\overline{a}_n) > \overline{w}g(b_1w,\dots,b_nw)$, then for any choice of $a_1,\dots,a_n\in\mathcal{O}_w$ with $a_iw=\overline{a}_i$ for $1\le i\le n$ we would obtain that $a_1,\dots,a_n\in\mathcal{O}_v$ and $vf(a_1,\dots,a_n) > vf(b_1,\dots,b_n)$, a contradiction. □

We use this lemma to prove the existence of the non-extremal fields in mixed characteristic as claimed in Proposition 1.12. We consider again the two non-extremal fields (L,v) and (L,w) mentioned above. By Theorem 2.14 of [9] there is an extension (K_0,v_0) of $(\mathbb{Q},v_p)$ with divisible value group and L as its residue field. We replace (K_0,v_0) by a maximal immediate extension (M,v_0). Then (M,v_0) is algebraically complete, and so are $(M,v_0\circ v)$ and $(M,v_0\circ w)$. The value group of $(M,v_0\circ v)$ is a $\mathbb{Z}$-group, and $(M,v_0\circ w)$ has divisible value group and nonperfect large residue field. But by Lemma 4.1, both fields are non-extremal.

Finally, we have to prove the existence of extremal fields as stated in parts c) and d) of Proposition 1.12. We will employ Theorem 1.13 which we will prove now. We note that by Theorem 1.1, the residue field of an extremal field with divisible value group must be large. Also, every non-trivially valued extremal field is henselian, which implies that it is itself a large field. Therefore, it remains to prove the following assertion:

Let (K,v) be any $\aleph_1$-saturated valued field. Assume that Γ and Δ are convex subgroups of vK such that $\Delta \subsetneqq \Gamma$ and Γ/Δ is archimedean. Let u (respectively w) be the coarsening of v corresponding to Δ (resp. Γ). Denote by $\bar{u}$ the valuation induced on Kw by u. Then $(Kw,\bar{u})$ is maximal and extremal, and its value group is isomorphic either to $\mathbb{Z}$ or to $\mathbb{R}$.

Proof. Denote by $\mathcal{O}_u$ (resp. $\mathcal{O}_w$) the valuation ring corresponding to u (resp. w). We note that the value group of u is vK/Δ, the value group of w is vK/Γ, and we have that

$$\mathcal{O}_v \;\subset\; \mathcal{O}_u \;\subset\; \mathcal{O}_w\,.$$

We show first that $(Kw,\bar{u})$ is maximal. From [6, Theorem 4] we know that a valued field is maximal if and only if every pseudo Cauchy sequence has a limit in the field. We refer the reader to [6] for an excellent introduction to the theory of pseudo Cauchy sequences (which Kaplansky calls “pseudo-convergent sets”). If $(a_i)_{i<\lambda}$ is any pseudo Cauchy sequence in $(Kw, \bar{u})$, then the sequence $\bar{u}(a_{i+1} - a_i)$ in $\bar{u}(Kw) = \Gamma/\Delta$ is strictly increasing. But the cofinality of any strictly increasing sequence in $\mathbb{R}$ (and hence also in any archimedean ordered abelian group) is at most ω. Therefore, it suffices to show that every pseudo Cauchy sequence $(a_i)_{i<\omega}$ in $(Kw, \bar{u})$ has a limit. By definition, $a \in Kw$ is a limit of this sequence if and only if $\bar{u}(a - a_i) = \bar{u}(a_{i+1} - a_i)$ for all $i < \omega$.

Write $a_i = b_i w$ with $b_i \in \mathcal{O}_w$, $i < \omega$. Then the sequence $(u(b_{i+1} - b_i))_{i<\omega}$ is strictly increasing in vK/Δ. This implies that the sequence $(v(b_{i+1} - b_i))_{i<\omega}$ is strictly increasing in vK. We consider the following (partial) type in countably many parameters:

$$\{v(x - b_i) = v(b_{i+1} - b_i) \mid i < \omega\} .$$

It is finitely realizable in (K, v) since for $x = b_{i+1}$ we obtain that $v(x - b_j) = v(b_{j+1} - b_j)$ holds for $0 \leq j \leq i$. By saturation, there is some $b \in K$ which realizes this type. Now $v(b - b_i) = v(b_{i+1} - b_i)$ implies that

$$\begin{aligned} w(b - b_i) &= v(b - b_i) + \Gamma = v(b_{i+1} - b_i) + \Gamma = w(b_{i+1} - b_i) , \\ u(b - b_i) &= v(b - b_i) + \Delta = v(b_{i+1} - b_i) + \Delta = u(b_{i+1} - b_i) . \end{aligned}$$

The former implies that $b \in \mathcal{O}_w$ as also all b_i are in $\mathcal{O}_w$; so we can set $a := bw$. The latter then implies that

$$\bar{u}(a - a_i) = \bar{u}(bw - b_i w) = \bar{u}((b - b_i)w) = u(b - b_i) = u(b_{i+1} - b_i) = \bar{u}(a_{i+1} - a_i) .$$

This proves that $a \in Kw$ is a limit of the pseudo Cauchy sequence $(a_i)_{i<\omega}$ and shows that $(Kw, \bar{u})$ is maximal.

Now we distinguish two cases.

Case 1: $\bar{u}(Kw)$ is isomorphic to $\mathbb{Z}$. In this case, it follows from the maximality that $(Kw, \bar{u})$ is algebraically complete and hence extremal [2, by Theorem 4.1].

Case 2: $\bar{u}(Kw) = \Gamma/\Delta$ is densely ordered. Note that since the archimedean ordered group Γ/Δ is embeddable in $\mathbb{R}$, any subset of it has coinitiality and cofinality no greater than $\aleph_0$.

We show that $(Kw, \bar{u})$ is extremal. The value group $\bar{u}(Kw)$ is Γ/Δ and $\mathcal{O}_{\bar{u}}$ is the image of $\mathcal{O}_u$ under the residue map $x \mapsto xw$ of w. For a tuple $\mathbf{a} = (a_1, ..., a_m)$ from $\mathcal{O}_w$, we denote by $\mathbf{a}w := (a_1 w, ..., a_m w)$ the corresponding tuple of residues.

Let $\bar{f} \in Kw[\mathbf{x}]$ be a polynomial in the variables $\mathbf{x} = (x_1, ..., x_m)$ and let $f \in \mathcal{O}_w[\mathbf{x}]$ denote any lift of $\bar{f}$ so that $fw = \bar{f}$. We must show that the set of $\bar{u}$-values of the image of $\bar{f}$, i.e.,

$$\begin{aligned} X &:= \left\{\bar{u}(\bar{f}(\mathbf{b})) \in \Gamma/\Delta \cup \{\infty\} \mid \mathbf{b} \in \mathcal{O}_{\bar{u}} \right\} \\ &= \left\{\bar{u}(\bar{f}(\mathbf{a}w)) \in \Gamma/\Delta \cup \{\infty\} \mid \mathbf{a} \in \mathcal{O}_u \right\} , \end{aligned}$$

has a maximum. As noted above, the cofinality of X is no greater than $\aleph_0$. Thus there is a sequence $(\mathbf{a}_n)_{n<\omega}$ of m-tuples from $\mathcal{O}_u$ such that the sequence

$$(\bar{u}(\bar{f}(\mathbf{a}_n w))_{n<\omega}$$

is increasing and cofinal in X. For each $n < \omega$ we set $\alpha_n := v(f(\mathbf{a}_n))$, and note that either $\bar{f}(\mathbf{a}_n w) = 0$ (in which case $\bar{u}(\bar{f}(\mathbf{a}_n w)) = \infty$ must be the maximum of X) or

$$\alpha_n + \Delta = u(f(\mathbf{a}_n)) = \bar{u}(\bar{f}(\mathbf{a}_n w)).$$

Next we set $Y := \{\gamma + \Delta \in \Gamma/\Delta \mid \gamma + \Delta < \Delta\}$. Then Y is equal to the image under u of the elements of $\mathcal{O}_w \setminus \mathcal{O}_u$ (and also equal to the image under $\bar{u}$ of $Kw \setminus \mathcal{O}_{\bar{u}}$). By assumption, Γ/Δ is densely ordered; thus Y has no maximum. Also the cofinality of Y can be no greater than $\aleph_0$. Thus there is a sequence $(\beta_n)_{n<\omega}$ in Γ such that $(\beta_n + \Delta)_{n<\omega}$ is a strictly increasing and cofinal sequence in Y.

Finally we consider the following (partial) $\mathbf{x}$-type in countably many parameters:

$$p(\mathbf{x}) := \{\alpha_n \le v(f(\mathbf{x})) \mid n < \omega\} \cup \{\beta_n \le v(x_i) \mid n < \omega, 1 \le i \le m\} \ .$$

This is finitely realised in (K, v). By saturation, it is realized by some m-tuple $\mathbf{c} = (c_1, ..., c_m) \in K^m$.

For $1 \le i \le m$ we examine the second set of formulas in $p(\mathbf{x})$ to find that $\beta_n \le v(c_i)$, for each $n < \omega$. Thus $\beta_n + \Delta \le v(c_i) + \Delta = u(c_i)$, again for each $n < \omega$. By the cofinality of the sequence $(\beta_n + \Delta)_{n<\omega}$ in Y we have that $c_i \in \mathcal{O}_u$.

Finally, by examining the first set of formulas in $p(\mathbf{x})$, we see that $\alpha_n \le v(f(\mathbf{c}))$, for all $n < \omega$. Then either $\bar{u}(\bar{f}(\mathbf{c}w)) = \infty$ (in which case ∞ is the maximum of X) or we have that

$$\alpha_n + \Delta \ \le \ v(f(\mathbf{c})) + \Delta \ = \ u(f(\mathbf{c})) \ = \ \bar{u}(\bar{f}(\mathbf{c}w)),$$

for all $n < \omega$. Since $(\alpha_n + \Delta)$ is cofinal in X, $\bar{u}(\bar{f}(\mathbf{c}w))$ is the maximum of X. This shows that $(Kw, \bar{u})$ is extremal, as required.

For the conclusion of the proof, we show that the value group of $(Kw, \bar{u})$ is cut complete, which shows that it is isomorphic to $\mathbb{R}$. Take a Dedekind cut (D, E) in $\bar{u}(Kw)$, that is, D is a nonempty initial segment of $\bar{u}(Kw)$ and E is a nonempty final segment of $\bar{u}(Kw)$ such that $D \cup E = \bar{u}(Kw)$. As noted before, the cofinality of D and the coinitiality of E are no greater than $\aleph_0$. Thus there are sequences $(\beta_n)_{n<\omega}$ and $(\gamma_n)_{n<\omega}$ in Γ such that $(\beta_n + \Delta)_{n<\omega}$ is an increasing and cofinal sequence in D and $(\gamma_n + \Delta)_{n<\omega}$ is a decreasing and coinitial sequence in E. We consider the following (partial) type in countably many parameters:

$$\{\beta_n \le vx \mid n < \omega\} \cup \{\gamma_n \ge vx \mid n < \omega\} \ .$$

This is finitely realized in (K, v). Hence by saturation, it is realized by some $d \in K$. Then $\beta_n \le vd \le \gamma_n$ and therefore $\beta_n + \Delta \le ud \le \gamma_n + \Delta$, for each $n < \omega$. It follows that ud lies in the convex hull of Γ/Δ in vK/Δ, which shows that $wd = 0$. So $dw \in Kw$, and we obtain that

$$D \le \bar{u}(dw) = ud \le E \ ,$$

which proves that the cut (D, E) is realized in $\bar{u}(Kw)$, showing that this group is cut complete. □

We may choose (K, v) so that Γ/Δ is densely ordered, for any $\Delta \subset \Gamma \subset vK$. Indeed, if we take any integer $n \ge 2$ and (K, v) such that vK is n-divisible, then also Γ/Δ will be n-divisible and hence densely ordered. If on the other hand, the residue field Kv is imperfect and $w \subset u \subset v$ are as in the theorem, then also the residue field of $(Kw, \bar{u})$, which is equal to Ku, is imperfect. Taking (K, v) to be an $\aleph_1$-saturated valued field of equal characteristic p with imperfect large residue field and n-divisible value group, and choosing Γ and Δ according to Remark 1.14, we obtain from Theorem 1.13:

Corollary 4.2. *Let p be a prime. There exist extremal fields of equal characteristic p with value group isomorphic to $\mathbb{R}$ and imperfect residue field.*

To give an example of an extremal field obtained by this corollary, we begin by taking any $\aleph_1$-saturated elementary extension (K,v) of the Puiseux series field $\bigcup_{n\in\mathbb{N}}\mathbb{F}_p(x)((t^{1/n}))$ over $\mathbb{F}_p(x)$, where x is transcendental over $\mathbb{F}_p$. As the residue field Kv is an elementary extension of the lower residue field, it is also imperfect. As the value group vK is an elementary extension of the lower value group, it is also divisible.

On the other hand, we can extend the p-adic valuation from $\mathbb{Q}$ to a valuation v on $\mathbb{Q}(x)$ such that xv is transcendental over $\mathbb{F}_p$; then the residue field of $(\mathbb{Q}(x),v)$ will be the imperfect field $\mathbb{F}_p(xv)$. By adjoining n-th roots repeatedly, we can pass, without changing the residue field, to an algebraic extension (k,v) of $(\mathbb{Q}(x),v)$ with n-divisible value group. Now we can take any $\aleph_1$-saturated elementary extension (K,v) of (k,v). Then Kv will again be imperfect, vK will be n-divisible, and (K,v) will have mixed characteristic $(0,p)$.

In order to achieve that the valued field $(Kw,\bar{u})$ in Theorem 1.13 also has mixed characteristic, we choose Γ and Δ as follows. We take Δ to be the largest convex subgroup of vK not containing vp and let Γ be the smallest convex subgroup of vK containing vp. Then Δ is the largest proper convex subgroup of Γ, and therefore Γ/Δ is archimedean. It follows that $pw\neq 0$ since $vp\notin\Delta$, but $(pw)\bar{u}=pu=0$ since $vp\in\Gamma$. This shows that $\mathrm{char}\,Kw=0$ and $\mathrm{char}\,(Kw)\bar{u}=p$. We thus obtain:

Corollary 4.3. *Let p be a prime. There exist extremal fields of mixed characteristic $(0,p)$ with value group isomorphic to $\mathbb{R}$ and imperfect residue field.*

Corollaries 4.2 and 4.3 together complete the proof of Proposition 1.12.

By taking (K,v) as in one of these corollaries and (L,v) to be a countable model of $\mathrm{Th}\,(K,v)$, we obtain:

Corollary 4.4. *Let p be a prime. There exist countable extremal fields of equal characteristic p with divisible value group not isomorphic to $\mathbb{R}$ and imperfect residue field. Likewise, there exist countable extremal fields of mixed characteristic $(0,p)$ with divisible value group not isomorphic to $\mathbb{R}$ and imperfect residue field.*

By choosing models of arbitrary cardinality, one can obtain divisible value groups of arbitrarily large cardinality. But we do not know which divisible ordered abelian groups (and not even which cardinalities) can be thus obtained, as we are lacking an AKE-principle.

We will now give the
Proof of Corollary 1.15: We take (K,v) to be an $\aleph_1$-saturated elementary extension of an arbitrary non-large valued field whose value group is divisible by some $n\geq 2$, and apply Theorem 1.13. Since also vK is divisible by n, for all u and w as in the theorem the value group of $(Kw,\bar{u})$ is divisible. Hence if $v=w_1\circ w_2\circ w_3$ with w_2 of rank 1, then by setting $w=w_1$ and $u=w_1\circ w_2$ it follows from the theorem that (Kw_1,w_2) is extremal with nontrivial divisible value group, hence
a) w_2 is henselian,
b) Kw_1 is large,
c) $(Kw_1)w_2$ is large. □

For the conclusion of this paper, let us discuss how the property of extremality behaves in a valued field extension $(L|K,v)$ where (K,v) is existentially closed in (L,v). In this case, it is known that $L|K$ and $Lv|Kv$ are regular extensions and that vL/vK is torsion free. (An extension $L|K$ of fields is called **regular** if it is separable and K is relatively algebraically closed in L.)

Proposition 4.5. *Take a valued field extension $(L|K,v)$ such that (K,v) is existentially closed in (L,v), a subset S_K of K that is existentially definable with parameters in K, and a polynomial f in n variables over K. Denote by S_L the subset of L defined by the existential formula that defines S_K in K. Then the following assertions hold.*
a) If (K,v) is S_K-extremal w.r.t. f, then (L,v) is S_L-extremal w.r.t. f and $\max vf(S_L^n) = \max vf(S_K^n)$. In particular, if (K,v) is extremal, then (L,v) is extremal w.r.t. all polynomials with coefficients in K.
b) Assume in addition that $vL = vK$. If (L,v) is S_L-extremal w.r.t. f, then (K,v) is S_K-extremal w.r.t. f and $\max vf(S_L^n) = \max vf(S_K^n)$. In particular, if (L,v) is extremal, then so is (K,v).

Proof. a): Assume that $a \in S_K^n$ such that $vf(a) = \max vf(S_K^n)$. Then the assertion that there exists an element b in S_L^n such that $vf(b) > vf(a)$ is an elementary existential sentence with parameters in K. Hence if it held in L, then there would be an element b' in S_K^n such that $vf(b') > vf(a)$, which is a contradiction to the choice of a. It follows that $\max vf(S_L^n) \leq \max vf(S_K^n)$. Since $S_K \subseteq S_L$, we obtain that $\max vf(S_L^n) = \max vf(S_K^n)$.
b): Take $b \in S_L^n$ such that $vf(b) = \max vf(S_L^n)$. Since $vL = vK$ by assumption, there is $c \in K$ such that $vc = vf(b)$. Now the assertion that there exists an element b in S_L^n such that $vf(b) = vc$ is an elementary existential sentence with parameters in K. Hence there is $a \in S_K^n$ such that $vf(a) = vc = \max vf(S_L^n)$. Since $vf(a) \in vf(S_K^n) \subseteq vf(S_L^n)$, we obtain that $vf(a) = \max vf(S_K^n)$. □

References

[1] Anscombe, S. – Fehm, A.: *The existential theory of equicharacteristic henselian valued fields*, http://arxiv.org/abs/1501.04522 (2015)
[2] Azgin, S. – Kuhlmann, F.-V. – Pop, F.: *Characterization of Extremal Valued Fields*, Proc. Amer. Math. Soc. **140** (2012), 1535–1547
[3] van den Dries, L. – Kuhlmann, F.-V.: *Images of additive polynomials in $\mathbb{F}_q((t))$ have the optimal approximation property*, Can. Math. Bulletin **45** (2002), 71–79
[4] Denef, J. – Schoutens, H.: *On the decidability of the existential theory of $F_p[[t]]$*, Valuation theory and its applications, Vol. II (Saskatoon, SK, 1999), 43?60, Fields Inst. Commun., 33, Amer. Math. Soc., Providence, RI, 2003
[5] Durhan, S.: *Additive Polynomials over Perfect Fields*, in: Valuation Theory in Interaction, Proceedings of the Second International Valuation Theory Conference, Segovia / ElEscorial 2011, EMS Series of Congress Reports 2014
[6] Kaplansky, I.: *Maximal fields with valuations I*, Duke Math. Journ. **9** (1942), 303–321
[7] Kuhlmann, F.-V.: *Quantifier elimination for henselian fields relative to additive and multiplicative congruences*, Israel J. Math. **85** (1994), 277–306
[8] Kuhlmann, F.-V.: *Elementary properties of power series fields over finite fields*, J. Symb. Logic, **66**, 771–791 (2001)
[9] Kuhlmann, F.-V.: *Value groups, residue fields and bad places of rational function fields*, Trans. Amer. Math. Soc. **356** (2004), 4559–4600
[10] Kuhlmann, F.-V.: *On places of algebraic function fields in arbitrary characteristic*, Advanves in Math. **188** (2004), 399–424
[11] Kuhlmann, F.–V.: *Additive polynomials and their role in the model theory of valued fields*, Logic in Tehran, Lecture Notes in Logic **26** 160–203, Assoc. Symbol. Logic, La Jolla, CA, 2006
[12] Kuhlmann, F.–V.: *Dense subfields of Henselian fields, and integer parts*, Logic in Tehran, Lecture Notes in Logic **26** 204?-226, Assoc. Symbol. Logic, La Jolla, CA, 2006

[13] Kuhlmann, F.-V.: *The algebra and model theory of tame valued fields*, to appear in: J. reine angew. Math. Preliminary version published in: Séminaire de Structures Algébriques Ordonnées, **81**, Prépublications Paris 7 (2009)

[14] Kuhlmann, F.-V.: *A classification of Artin-Schreier defect extensions and a characterization of defectless fields*, Illinois J. Math. **54** (2010), 397–448

[15] Pop, F.: *Embedding problems over large fields*, Annals of Math. **144** (1996), 1–34.

Jeremiah Horrocks Institute, Leighton Building Le7, University of Central Lancashire, Preston, PR1 2HE, United Kingdom

E-mail address: `sanscombe@uclan.ac.uk`

Institute of Mathematics, University of Silesia, ul. Bankowa 14, 40-007 Katowice, Poland

E-mail address: `fvk@math.usask.ca`