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# On convoluters on $L^p$-spaces

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**Abstract**

We prove two theorems about convolution operators on $L^p(G)$ for a locally compact group $G$. First, if $G$ has the approximation property, then the algebra of convoluters is the algebra of pseudo-measures. Second, the bicommutant of the algebra of pseudo-measures is the algebra of convoluters.

## 1 Introduction

Let $G$ be a locally compact group and $L^p(G)$ denote the $L^p$-space with respect to left Haar measure, for $p \in [1, \infty]$. The algebra of bounded operators $\mathcal{B}(L^p(G))$ for $p \in (1, \infty)$ admits a natural predual (which we specify in Section 2, below) and hence admits a specified weak* topology which is finer than the weak operator topology. We consider the two subalgebras, the *pseudo-measures* and *convoluters*, which are given by

$$PM_p(G) = \overline{\mathrm{lin}}^{w*} \lambda_p(G) \qquad \text{(weak* closed linear span)}$$
$$CV_p(G) = \rho_p(G)' \qquad \text{(commutant)}$$

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where the right and left regular representations $\lambda_p, \rho_p : G \to \mathcal{B}(L^p(G))$ are given by

$$\lambda_p(s)f(t) = f(s^{-1}t), \quad \rho_p(s)f(t) = \Delta(s)^{1/p}f(ts)$$

for all $s$ and a.e. $t$. Here $\Delta$ is the Haar modular function. Since $\lambda_p(G) \subseteq \rho_p(G)'$ and commutant algebras are weak operator closed, hence weak* closed, we have that $PM_p(G) \subseteq CV_p(G)$. Notice, moreover, that the invertible isometry $U$ in $\mathcal{B}(L^p(G))$, given by $Uf(t) = \Delta^{1/p}(t^{-1})f(t^{-1})$, is self-inverse and intertwines $\lambda_p$ and $\rho_p$, which shows we may interchange the roles of left and right.

The *approximation property* was defined by Haagerup and Kraus [9]. A detailed definition is provided in Section 3, below.

**Theorem 1.1.** *If $G$ has the approximation property, then*

$$CV_p(G) = PM_p(G).$$

A complete description of connected groups with the approximation property is provided by Haagerup, Knudby and de Laat [10]: any simple Lie quotient must have real rank 1. This builds on an enormous body of work, including [18, 11, 12]. The approximation property passes to closed subgroups, extension groups, and is passed up from lattices [9]. In particular, countably generated free groups enjoy this property, even the stronger property of *weak amenability*, defined by de Cannière and Haggerup [7]. The approximation property does not pass to general quotients: the finitely generated lattice $\mathrm{SL}_3(\mathbb{Z})$ in $\mathrm{SL}_3(\mathbb{R})$ is a quotient of some free group $\mathbb{F}_n$. In fact, we have the following implications of properties of $G$:

$$\text{amenable} \;\Rightarrow\; \text{weakly amenable} \;\Rightarrow\; \text{approximation property.}$$

The first implication is given by Leptin [19], i.e. a bounded approximate identity in the Fourier algebra $A_2(G)$ satisfies the definition of weak amenability of [7]. For amenable $G$, Theorem 1.1, is proved by Herz [15]; see Remark 2.9.

A certain $p$-approximation property, which is implied by the approximation property was defined by An, Lee and Ruan [1], and studied vigorously by Vergara [23]. Using this, Vergara strengthens our Theorem 1.1, but relies heavily on our methods. See Remark 3.3, below.

On a related note, we also give an elementary proof of the following. Recall that the intertwiner $U$ is given above

**Theorem 1.2.** *We have commutation results:*

$$PM_p(G)' = U\,CV_p(G)U \text{ and } (U\,CV_p(G)U)' = CV_p(G).$$

*In particular, we have bicommutant* $PM_p(G)'' = CV_p(G)$.

For $p = 2$ the comutation result of convoluters was proved by Dixmier [8] using left Hilbert algebras. Recently, Pham [21] gave an elementary proof in the $p = 2$ case. This proof is similar to the one offered here, but we wish to note that ours was conducted independently and posted on `arXiv` in 2016.

The present article is an updated version of our work [6], the main body of which was first posted to the `arXiv` in 2013. Both theorems appear to be folklore — see Cowling [3], and Herz [15] — but we have been unable to track down complete proofs. Theorem 1.1 was used in recent work of Öztop and the second named author [20] to obtain, for a pair of groups $G$ and $H$ with approximation property, a tensor product description of $A_p(G \times H)$. Given recent developments related to this work and interest in it, in particular [21, 23], we have elected to update our results and submit them for publication.

# 2 On preduals of pseudo-measures and convolvers

The entire goal of the present section is to present background for the proofs of our two main theorems. We also redevelop some methods of Cowling [3], who gave an innovative and insightful description of a predual of $CV_p(G)$. The present methods give a new perspective and help to illuminate the role of the Herz-Schur multipliers which we discuss in 2.2, below.

We let $p'$ be the conjugate index to $p$, so $\frac{1}{p} + \frac{1}{p'} = 1$. We recall that $\mathcal{B}(L^p(G))$ is the dual of the space $N^p(G) = L^{p'}(G)\hat{\otimes}L^p(G)$ by way of dual pairing $\langle T, \xi \otimes f\rangle = \langle \xi, Tf\rangle = \int_G \xi(t)[Tf](t)\,dt$. Elements $\omega = \sum_{n=1}^{\infty} \xi_n \otimes f_n$ of $N^p(G)$ may be viewed as functions $\omega(s,t) = \sum_{n=1}^{\infty} \xi_n(s) f_n(t)$ up to marginal almost everywhere (m.a.e.) equivalence; i.e. excepting marginally null sets, $(N_1 \times G) \cup (G \times N_2)$, where $N_1$ and $N_2$ are Haar null sets.

We let $P : N^p(G) \to A_p(G) \subseteq C_0(G)$ be given for $s$ in $G$ by

$$P\left(\sum_{n=1}^{\infty} \xi_n \otimes f_n\right)(s) = \sum_{n=1}^{\infty} \langle \xi_n, \lambda_p(s) f_n\rangle = \sum_{n=1}^{\infty} \xi_n * \check{f}_n$$

where $\check{f}(t) = f(t^{-1})$ a.e., i.e. $P\omega(s) = \int_G \omega(t, s^{-1}t)\,dt$. Hence $A_p(G)$ is normed as the coimage of $P$, i.e. $A_p(G) \cong N^p(G)/\ker P$ isometrically. It is straightforward to see that $PM_p(G)$ and $CV_p(G)$ admit the following pre-annihilators in $N^p(G)$:

$$\begin{aligned} {}^{\perp}PM_p(G) &= \ker P \\ {}^{\perp}CV_p(G) &= \overline{\mathrm{lin}}\{\rho_{p'}(t^{-1})\xi \otimes f - \xi \otimes \rho_p(t)f : \\ &\qquad\qquad t \in G, \xi \in L^{p'}(G), f \in L^p(G)\} \end{aligned}$$

where we note that adjoint of these right translation operators is given by $\rho_p(t)^* = \rho_{p'}(t^{-1})$.

It is standard that

$$PM_p(G) \cong A_p(G)^* \text{ and } CV_p(G) \cong [N^p(G)/{}^{\perp}CV_p(G)]^*$$

which gives us distinguished preduals of these algebras of operators.

## 2.1 $N^p(G)$ as an algebra

Herz [13] showed that $A_p(G)$ is always a subalgebra of $C_0(G)$, and further showed ([14]) that it has Gelfand spectrum given by evaluations at points of $G$. We note that $A_p(G)$ is known as the *Figà-Talamanca–Herz algebra*. Herz's technique for showing that $A_p(G)$ is an algebra lifts naturally to $N^p(G)$. We note that the amplification $T \mapsto T \otimes I : \mathcal{B}(L^p(G)) \to \mathcal{B}(L^p(G; L^p(G))) \cong \mathcal{B}(L^p(G \times G))$ is an isometry. We consider the *fundamental isometry* $W_p$ in $L^p(G \times G)$ given for a.e. $(s,t)$ by

$$W_p f(s,t) = f(s, st).$$

We note that $W_p$ is invertible with $W_p^* = W_{p'}^{-1}$. We define a *co-product* on $\mathcal{B}(L^p(G))$, $\Gamma : \mathcal{B}(L^p(G)) \to \mathcal{B}(L^p(G \times G))$ by

$$\Gamma(T) = W_p^{-1}(T \otimes I)W_p.$$

This spatially implemented map is evidently weak*-weak* continuous, and hence admits a preadjoint. We note that $N^p(G) \otimes N^p(G)$ comprises a dense subspace of $N^p(G \times G)$ given on elementary tensors of elementary tensors ("really elementary tensors") by $(\xi \otimes f) \otimes (\eta \otimes g) \mapsto (\xi \otimes \eta) \otimes (f \otimes g)$. Let us compute $\Gamma_*$ on these really elementary tensors. We have

$$\begin{aligned} \langle T, \Gamma_*((\xi \otimes f) \otimes (\eta \otimes g))\rangle &= \langle W_p^{-1}(T \otimes I)W_p, (\xi \otimes f) \otimes (\eta \otimes g)\rangle \\ &= \langle W_{p'}(\xi \otimes \eta), (T \otimes I)W_p(f \otimes g)\rangle \\ &= \int_G \int_G \xi(s)\eta(st)(T \otimes I)[(s,t) \mapsto f(s)g(st)]\, ds\, dt \\ &= \int_G \langle \xi\eta_t, T(fg_t)\rangle\, dt \end{aligned}$$

where $\eta_t(s) = \eta(st)$, for example. Hence we see that

$$\Gamma_*((\xi \otimes f) \otimes (\eta \otimes g)) = \int_G (\xi\eta_t) \otimes (fg_t)\, dt. \tag{2.1}$$

The equation

$$\int_G \int_G (\xi\eta_t\vartheta_s) \otimes (fg_t h_s)\, dt\, ds = \int_G \int_G (\xi\eta_t\vartheta_{ts}) \otimes (fg_t h_{ts})\, ds\, dt$$

shows that $\Gamma_*$ is an associative product. We note that though $\Gamma(\lambda_p(r)) = \lambda_p(r) \otimes \lambda_p(r)$, which implies that $\Gamma|_{PM_p(G)}$ is cocommutative, we have for a multiplication operator $M_\varphi$, where $\varphi \in L^\infty(G)$, that $\Gamma(M_\varphi) = M_\varphi \otimes I$, which shows that $\Gamma$ is not generally cocommutative. Hence $\Gamma_*$ is not commutative.

Let $\mathcal{K}(G)$ denote the family of all compact neighbourhoods of the identity in $G$. For $K$ in $\mathcal{K}(G)$ we let $L^p(K) = 1_K L^p(G)$, which is a 1-complemented subspace. Likewise we define $L^{p'}(K)$ and

$$N^p(K) = L^{p'}(K) \hat{\otimes} L^p(K)$$

which is a subspace of $N^p(G)$.

**Proposition 2.1.** *The space $^\perp PM_p(G) = \ker P$ is a $\Gamma_*$-ideal in $N^p(G)$, while each of $^\perp CV_p(G)$ and $N^p(K)$, for $K$ in $\mathcal{K}(G)$, are right ideals.*

*Proof.* That $P : N^p(G) \to C_0(G)$ is $\Gamma_*$-pointwise multiplicative is shown in [13], hence $^\perp PM_p(G) = \ker P$ is an ideal. Let us consider the case of $^\perp CV_p(G)$, manually. On really elementary tensors we have

$$\begin{aligned}
&\Gamma_*((\rho_{p'}(s^{-1})\xi \otimes f - \xi \otimes \rho_p(s)f) \otimes (\eta \otimes g)) \\
&\quad = \int_G [(\rho_{p'}(s^{-1})\xi)\eta_t \otimes fg_t - \xi\eta_t \otimes (\rho_p(s)f)g_t]\, dt \\
&\quad = \int_G \rho_{p'}(s^{-1})(\xi\eta_{st}) \otimes fg_t\, dt - \int_G \xi\eta_t \otimes \rho_p(s)(fg_{s^{-1}t})\, dt \\
&\quad = \int_G [\rho_{p'}(s^{-1})(\xi\eta_{st}) \otimes fg_t - \xi\eta_{st} \otimes \rho_p(s)(fg_t)]\, dt
\end{aligned}$$

which is an integral of elements in $^\perp CV_p(G)$. That each $N^p(K)$ is a right ideal follows readily from (2.1). Indeed if $\xi \otimes f \in N^p(K)$ then

$$\begin{aligned}
(1_K \otimes 1_K)\Gamma_*((\xi \otimes f) \otimes (\eta \otimes g)) &= \int_G 1_K\xi\eta_t \otimes 1_K fg_t\, dt \\
&= \Gamma_*((\xi \otimes f) \otimes (\eta \otimes g))
\end{aligned}$$

so $\Gamma_*((\xi \otimes f) \otimes (\eta \otimes g)) \in N^p(K)$. $\square$

It will be convenient below, to consider the following space of elements. We let $L^1_{\mathrm{loc}}(G)$ denote the space of locally a.e. equivalence classes of measurable functions on $G$ which are integrable on any compact set. An application of Hölder's inequality shows that $L^q(G) \subseteq L^1_{\mathrm{loc}}(G)$ for any $q$ in $[1, \infty]$. If $h \in L^1_{\mathrm{loc}}(G)$ and $g \in C_c(G)$ then $h * g$ is well-defined as an element of $L^1_{\mathrm{loc}}(G)$. We let

$$CV_p^1(G) = \big\{h \in L^1_{\mathrm{loc}}(G) : \sup\{\|h * g\|_p : g \in C_c(G),\ \|g\|_p \leq 1\} < \infty\big\}.$$

Each element $h$ of $CV_p^1(G)$ defines an operator $\lambda_p(h)$ in $\mathcal{B}(L^p(G))$. By testing on elements $\rho_{p'}(s^{-1})g \otimes f - g \otimes \rho_p(s)f$ where $f, g \in C_c(G)$, and applying a standard density argument, we see that $\lambda_p(h) \in CV_p(G)$.

The next result is really a repackaging of the main result of Cowling [3], and is used extensively through the rest of this note. The second part is a localization theorem.

**Theorem 2.2.** (i) *Each $T$ in $CV_p(G)$ may be approximated in the strong operator topology by a bounded net of elements $\lambda_p(h_i)$ where each $h_i \in CV_p^p(G) = CV_p^1 \cap L^p(G)$.*

(ii) *For each $K$ in $\mathcal{K}(G)$, we have that*

$$^{\perp}PM_p(G) \cap N^p(K) = {}^{\perp}CV_p(G) \cap N^p(K).$$

*Proof.* (i) Let $T \in CV_p(G)$. Let $(f_i) \subset C_c(G)$ be a contractive approximate identity for $L^1(G)$. Then for $g$ in $C_c(G)$ we have

$$T(f_i * g) = T(\rho_p(\Delta^{-1/p}\check{g})f_i) = \rho_p(\Delta^{-1/p}\check{g})T(f_i) = T(f_i) * g$$

where $\check{g}(t) = g(t^{-1})$ a.e. Notice then, that

$$\begin{aligned} &\|T(f_i) * g\|_p = \|T(f_i * g)\|_p \leq \|T\|\|f_i * g\|_p \leq \|T\|\|g\|_p \\ &\text{and } \|T(f_i) * g - Tg\|_p = \|T(f_i * g - g)\|_p \xrightarrow{i} 0. \end{aligned}$$

It follows that $h_i = T(f_i)$ gives the desired net of elements in $CV_p^p(G)$.

(ii) Since $^{\perp}PM_p(G) \supseteq {}^{\perp}CV_p(G)$, it suffices to show that $^{\perp}PM_p(G) \cap N^p(K) \subseteq {}^{\perp}CV_p(G)$. From (i), above, it suffices to show that for any $h$ in $CV_p^1(G)$, that $\lambda_p(h)$ annihilates $^{\perp}PM_p(G) \cap N^p(K)$.

Now let $\xi \otimes f \in N^p(K)$, and let $(f_n)$ be a sequence in $C_c(G)$ whose supports are in the neighbourhood $KK^{-1}$ of $K$ and converges to $f$ in $L^p(G)$.

Then we have that

$$\begin{aligned}\langle \xi, \lambda_p(h)f\rangle &= \lim_n \langle \xi, \lambda_p(h)f_n\rangle = \lim_n \int_K \xi(t) h * f_n(t)\, dt \\ &= \lim_n \int_K \xi(t) \int_{K^2K^{-1}} h(s) f_n(s^{-1}t)\, ds\, dt \\ &= \lim_n \int_{K^2K^{-1}} h(s) \int_K \xi(t) f_n(s^{-1}t)\, dt\, ds \\ &= \lim_n \int_{K^2K^{-1}} h(s) P(\xi \otimes f_n)(s)\, ds \\ &= \int_{K^2K^{-1}} h(s) P(\xi \otimes f)(s)\, ds\end{aligned}$$

where each interchange of integrals is justified by Fubini's theorem, as $(\xi \otimes h)1_{K\times K^2K^{-1}}$ is integrable and each $f_n$ is bounded, and the limits are justified by $L^p$-convergence of $f_n$, then uniform convergence of $P(\xi \otimes f_n)$. Hence, if $\omega = \sum_{n=1}^{\infty} \xi_n \otimes f_n \in {}^{\perp}PM_p(G) \cap N^p(K)$, then uniform convergence of $\sum_{n=1}^{\infty} P(\xi_n \otimes f_n) = P\omega = 0$ provides that

$$\langle \lambda_p(h), \omega\rangle = \sum_{n=1}^{\infty} \langle \xi_n, \lambda_p(h) f_n\rangle = \sum_{n=1}^{\infty} \int_{K^2K^{-1}} h(s) P(\xi_n \otimes f_n)(s)\, ds = 0$$

which is the desired annihilation condition. □

We note that proving Theorem 1.1 amounts, in effect, to showing that $\lambda_p(CV_p^1(G)) \subseteq PM_p(G)$, even that $\lambda_p(CV_p^p(G)) \subseteq PM_p(G)$. Attacking this directly seems very delicate; see Cowling's approach [2] for certain simple connected Lie groups with real rank 1. We will proceed through a different sequence of observations.

**Corollary 2.3.** (i) *The space* $\bigcup_{K\in\mathcal{K}(G)} {}^{\perp}PM_p(G) \cap N^p(K)$ *is dense in* ${}^{\perp}CV_p(G)$.

(ii) *Every* $\Gamma_*$*-commutator is an element of* ${}^{\perp}CV_p(G)$. *Hence* ${}^{\perp}CV_p(G)$ *is a two-sided* $\Gamma_*$*-ideal.*

*Proof.* (i) We observe that the family of elements

$$\bigcup_{K\in\mathcal{K}(G)} \{\rho_{p'}(s^{-1})\xi \otimes f - \xi \otimes \rho_p(s) f : s \in K, \xi \in L^{p'}(K), f \in L^p(K)\}$$

is contained in $\bigcup_{K\in\mathcal{K}(G)} {}^{\perp}CV_p(G) \cap N^p(KK^{-1} \cup K^2)$, and its linear span is dense in ${}^{\perp}CV_p(G)$.

(ii) Any commutator $\Gamma_*(\omega\otimes\mu-\mu\otimes\omega)$ can be approximated by elements of the form $\Gamma_*([1_K\otimes 1_K]\omega\otimes[1_K\otimes 1_K]\mu-[1_K\otimes 1_K]\mu\otimes[1_K\otimes 1_K]\omega)$ where $K\in\mathcal{K}(G)$. Such elements are in ${}^\perp PM_p(G)\cap N^p(K)$, hence in ${}^\perp CV_p(G)$.

Now, if $\omega\in{}^\perp CV_p(G)$ and $\mu\in N^p(G)$ we have

$$\Gamma_*(\mu\otimes\omega)=\Gamma_*(\omega\otimes\mu)+\Gamma_*(\mu\otimes\omega-\omega\otimes\mu)\in{}^\perp CV_p(G)$$

which establishes the second fact. □

Hence $\Gamma_*$ induces a commutative product on

$$\overline{A_p}(G)=N^p(G)/{}^\perp CV_p(G)$$

which is a predual of $CV_p(G)$. We let $\overline{P}:N^p(G)\to\overline{A_p}(G)$ denote the quotient map. We note that there is a further natural quotient map $\overline{P}(\omega)\mapsto P\omega:\overline{A_p}(G)\to A_p(G)$. It is shown by Cowling [3] that this is the Gelfand transform. In particular, semisimplicty of $\overline{A_p}(G)$ is equivalent to having ${}^\perp CV_p(G)={}^\perp PM_p(G)$, i.e. $CV_p(G)=PM_p(G)$.

Now we let for $K$ in $\mathcal{K}(G)$

$$A_p(K)=P(N^p(K)).$$

We remark that if $K$ is an open subgroup of $G$, then our definition of $A_p(K)$ coincides with the usual definition. For $a$ in $A_p(K)$

$$\|a\|_{A_p(K)}=\inf\{\|\omega\|_{N^p(G)}:\omega\in N^p(K),\ a=P(\omega)\}$$

is a norm on $A_p(K)$.

The set $\bigcup_{K\in\mathcal{K}(G)}A_p(K)$ is the algebra $A_{p,c}(G)$ of compactly supported elements of $A_p(G)$. Indeed, if $u$ in $A_{p,c}(G)$ is supported on compact $S$, let $L\in\mathcal{K}(G)$ and $u_L=\frac{1}{m(L)}1_{SL}*\check{1}_L$ is 1 on $S$, hence $u=uu_L\in A_p(K)$ whenever $SL\cup L\subseteq K$.

The following will play a critical role in the proof of Lemma 3.2, and hence in the proof of Theorem 1.1.

**Corollary 2.4.** *Let $N^p_c(G)=\bigcup_{K\in\mathcal{K}}N^p(K)$. Then $\overline{P}(N^p_c(G))$ is a dense ideal in $\overline{A_p}(G)$ which is algebraically isomorphic to $A_{p,c}(G)$. Furthermore, for $\omega$ in $N^p_c(G)$ we have*

$$\|\overline{P}(\omega)\|_{\overline{A_p}(G)}=\inf\{\|P(\omega)\|_{A_p(K)}:\omega\in N^p(K)\}$$

*and the pairing with $CV_p(G)$ depends only on $P(\omega)$; i.e. if $T\in CV_p(G)$ we may write*

$$\langle T,\omega\rangle=\langle T,P\omega\rangle. \tag{2.2}$$

*Proof.* It is evident that $N_c^p(G)$ is a dense right $\Gamma_*$-ideal in $N^p(G)$, so $\overline{P}(N_c^p(G))$ is a dense ideal in $\overline{A_p}(G)$, by part (ii) of the prior corollary. We use part (i) of the prior corollary to see that

$$\begin{aligned}\|\overline{P}(\omega)\|_{\overline{A_p}(G)} &= \mathrm{dist}(\omega, {}^{\perp}CV_p(G)) \\ &= \inf_{K\in\mathcal{K}(G)} \mathrm{dist}(\omega, {}^{\perp}PM_p(G)\cap N^p(K)) \\ &= \inf\{\|P(\omega)\|_{A_p(K)} : K\in\mathcal{K}(G),\ \omega\in N^p(K)\}\end{aligned}$$

where we note that the norm on $A_p(K)$ is given as the coimage: $A_p(K)\cong N^p(K)/\ker P|_{N^p(K)}$, where $\ker P|_{N^p(K)} = {}^{\perp}PM_p(G)\cap N^p(K)$. Hence if, further, $P\omega = 0$ then for $T$ in $CV_p(G)$ we have

$$|\langle T,\omega\rangle| \leq \|T\|\|\bar{P}(\omega)\|_{\overline{A_p}(G)} = 0$$

which establishes (2.2). □

We record the following localization principle. In terminology of Herz [15], (b), below, is the condition of being "formally of compact support".

**Theorem 2.5.** *The following are equivalent:*

(a) $CV_p(G) = PM_p(G)$;

(b) *for each $u$ in $A_{p,c}(G)$ we have*

$$\|u\|_{A_p(G)} = \inf\{\|u\|_{A_p(K)} : K\in\mathcal{K}(G),\ u\in A_p(K)\};\ \textit{and}$$

(c) *there is $C>0$ such that for each $u$ in $A_{p,c}(G)$ we have*

$$\inf\{\|u\|_{A_p(K)} : K\in\mathcal{K}(G),\ u\in A_p(K)\} \leq C\|u\|_{A_p(G)}.$$

*Proof.* That (a) implies (b) is immediate form the last corollary, whilst that (b) implies (c) is obvious for $C\geq 1$. If (c) holds, then on the dense subspace $A_{p,c}(G)$, which we may identify with $\overline{P}(N_c^p(G))$, we have that $\|\cdot\|_{A_p(G)}$ and $\|\cdot\|_{\overline{A_p}(G)}$ are equivalent norms, so the map $\omega + {}^{\perp}CV_p(G)\mapsto P(\omega)$ is injective, which implies that ${}^{\perp}CV_p(G) = {}^{\perp}PM_p(G)$. The latter fact gives (a). □

## 2.2 The action of Herz-Schur multipliers

Let $[\mathrm{SQ}_p]$ denote the class of spaces which are isometrically isomorphic to a subspace of a quotient of an $L^p$-space. Kwapień [17] noted that for $E$ in $[\mathrm{SQ}_p]$ that the ampliation $T\mapsto T\otimes I_E : \mathcal{B}(L^p(G))\to\mathcal{B}(L^p(G;E))$ is an isometry. In fact, he showed that this property characterizes elements of the class $[\mathrm{SQ}_p]$. See Dales et al [4] for an exposition on this. Hence, by duality, the map $\gamma_E : L^{p'}(G; E^*)\hat{\otimes}L^p(G; E) \to N^p(G)$ given on elementary tensors m.a.e. by

$$\gamma_E(\Xi \otimes F)(s,t) = \langle \Xi(s), F(t)\rangle_{E^*,E} \tag{2.3}$$

is a quotient map, in particular a contraction. Also see Herz [14]. Let us define the *p-Herz-Schur multiplier* space by

$$\begin{aligned}\mathcal{M}_p(G) = \{\varphi : G \to \mathbb{C} \mid \varphi(st^{-1}) = \langle\beta(s), \alpha(t)\rangle \text{ where each}\\ \alpha : G \to E, \beta : G \to E^* \text{ is continuous}\\ \text{and bounded for some } E \text{ in } [\mathrm{SQ}_p]\}\end{aligned}$$

These were essentially described by Herz [15, 16], and the specific version we are using here was given by the first named author [5]. As shown in [5], $\mathcal{M}_p(G)$ is the algebra of "$p$-completely bounded multipliers" of $A_p(G)$, and a Banach algebra with respect to the norm

$$\|\varphi\|_{\mathcal{M}_p} = \inf\{\|\beta\|_\infty\|\alpha\|_\infty : \varphi(st^{-1}) = \langle\beta(s), \alpha(t)\rangle \text{ as above}\}.$$

**Proposition 2.6.** (i) *$N^p(G)$ is a Banach $\mathcal{M}_p(G)$-module via the action given m.a.e. by*

$$\varphi \cdot \omega(s,t) = \varphi(st^{-1})\omega(s,t).$$

(ii) *Each of ${}^\perp PM_p(G)$ and $N^p(K)$ ($K \in \mathcal{K}(G)$) are $\mathcal{M}_p(G)$-submodules, hence so too is ${}^\perp CV_p(G)$.*

*Proof.* (i) This is an immediate application of the map $\gamma_E$ in (2.3), above. If $\varphi \in \mathcal{M}_p(G)$ with $\varphi(st^{-1}) = \langle\beta(s), \alpha(t)\rangle$, and $\xi \otimes f$ is an elementary tensor in $N^p(G)$, then let $\xi\beta(s) = \xi(s)\beta(s)$ in $L^{p'}(G; E^*)$ and $f\alpha(t) = f(t)\alpha(t)$ in $L^p(G; E)$. Notice that $\|\xi\beta\|_{p';E^*} \le \|\xi\|_{p'}\|\beta\|_\infty$ and $\|f\alpha\|_{p;E} \le \|f\|_p\|\alpha\|_\infty$. Then

$$\varphi \cdot (\xi \otimes f) = \gamma_E(\xi\beta \otimes f\alpha) \in N^p(G)$$

and hence

$$\|\varphi \cdot (\xi \otimes f)\|_{N^p} \le \|\xi\beta\|_{p';E^*}\|f\alpha\|_{p;E} \le \|\beta\|_\infty\|\alpha\|_\infty\|\xi\|_{p'}\|f\|_p$$

from which it follows that $\|\varphi \cdot (\xi \otimes f)\|_{N^p} \le \|\varphi\|_{\mathcal{M}_p}\|\xi \otimes f\|_{N^p}$.

(ii) If $\varphi \in \mathcal{M}_p(G)$ and $\omega \in \ker P = {}^\perp PM_p(G)$ then

$$P(\varphi \cdot \omega)(s) = \int_G \varphi(t(s^{-1}t)^{-1})\omega(t, s^{-1}t)\,dt = \varphi(s)P\omega(s) = 0.$$

It is facile that each $N^p(K)$ is an $\mathcal{M}_p(G)$-submodule. Thus each ${}^{\perp}PM_p(G)\cap N^p(K)$ is an $\mathcal{M}_p(G)$-submodule, and the result for ${}^{\perp}CV_p(G)$ follows from the density result, Corollary 2.3 (i), above. $\square$

It is immediate that the action in (ii), above, induces actions of $\mathcal{M}_p(G)$ on each of $A_p(G)$ and $A_p(K)$ (each by pointwise multiplication) and on $\overline{A_p}(G)$, making each a continuous module. The spaces $A_p(K)$ are shown to be Banach $A_p(G)$-modules in [20], by a method quite similar to that above. We note that the natural map $\overline{P}(\omega)\mapsto P\omega : \overline{A_p}(G)\to A_p(G)$ is an $\mathcal{M}_p(G)$-module map.

We shall have need to consider the adjoint action on the dual spaces. That $CV_p(G)$ is a $\mathcal{M}_p(G)$-module seems also to be shown by Herz [16].

**Corollary 2.7.** *The algebras $CV_p(G)$ and $PM_p(G)$ are dual Banach $\mathcal{M}_p(G)$-modules:*

$$\langle \varphi\cdot T,\omega\rangle = \langle T,\varphi\cdot\omega\rangle.$$

*If $\varphi\in\mathcal{M}_p(G)$ and $h\in CV_p^1(G)$, then $\varphi h\in CV_p^1(G)$ and*

$$\varphi\cdot\lambda_p(h)=\lambda_p(\varphi h).$$

*Proof.* The first statement being an evident adjoint module operation, we are left only to inspect the second statement. However, if $\xi\otimes f$ is an elementary tensor in $N_c^p(G)$, i.e. in $N^p(K)$ for some $K$, then arguments just as in the proof of Theorem 2.2 (ii) provide that

$$\begin{aligned}\langle \varphi\cdot\lambda_p(h),\xi\otimes f\rangle &= \langle \lambda_p(h),\varphi\cdot(\xi\otimes f)\rangle\\ &= \int_{KK^{-1}} h(s)P(\varphi\cdot(\xi\otimes f))(s)\,ds\\ &= \int_{KK^{-1}} \varphi(s)h(s)\int_K \xi(t)f(s^{-1}t)\,dt\,ds\\ &= \langle \xi,(\varphi h)*f\rangle\end{aligned}$$

where the last interchange of integrals is justified as

$$(s,t)\mapsto |\varphi(s)h(s)\xi(t)f(s^{-1}t)|$$

is integrable thanks to Hölder's inequality and Tonelli's theorem. If $f\in C_c(G)$ then

$$\begin{aligned}\|(\varphi h)*f\|_p &= \sup\{|\langle\varphi\cdot\lambda_p(h),\xi\otimes f\rangle| : \xi\in L^p(K), K\in\mathcal{K}(G), \|\xi\|_p\le 1\}\\ &\le \|\varphi\cdot\lambda_p(h)\|\|f\|_p.\end{aligned}$$

Then taking supremum over such $f$ with $\|f\|_p\le 1$, reveals that $\varphi h\in CV_p^1(G)$, and further gives the desired formula. $\square$

We note that $A_p(G) \subseteq \mathcal{M}_p(G)$, is an ideal within the latter space, and the inclusion is a contractive map. Indeed if $a = \sum_{n=1}^{\infty} \langle \xi_n, \lambda_p(\cdot) f_n \rangle$ in $A_p(G)$, with $\sum_{n=1}^{\infty} \|\xi_n\|_{p'} \|f_n\|_p < \infty$ and each $\|\xi_n\|_{p'} \|f_n\|_p > 0$, let $\xi_n' = \|\xi_n\|_{p'}^{-1/p} \|f_n\|_p^{1/p'} \xi_n$ and $f_n' = \|\xi_n\|_{p'}^{1/p} \|f_n\|_p^{-1/p'} f_n$. Then $\beta(s) = (\lambda_{p'}(s^{-1}) \xi_n')_{n=1}^{\infty}$ in $\ell^{p'}(\mathbb{N}, L^{p'}(G))$ and $\alpha(t) = (\lambda_p(t^{-1}) f_n')_{n=1}^{\infty}$ in $\ell^p(\mathbb{N}, L^p(G))$ show that $a \in \mathcal{M}_p(G)$, with $\|a\|_{\mathcal{M}_p} \leq \|a\|_{A_p(G)}$.

Hence we get the following crucial result which will help us in the proof of Theorem 1.1.

**Corollary 2.8.** *Let $T \in CV_p(G)$ and $a \in A_p(G)$. Then $a \cdot T \in PM_p(G)$.*

*Proof.* By a norm estimate, we may assume that $a \in A_{p,c}(G)$. Furthermore, Theorem 2.2 (i) provides that $T$ is a weak* limit of $\lambda_p(h_i)$ for some $h_i$ in $CV_p^1(G)$. Then the prior corollary provides that each $a \cdot \lambda_p(h_i) = \lambda_p(ah_i)$, where $ah_i \in L^1_{\mathrm{loc}}(G)$ with compact support, hence $ah_i \in L^1(G)$, so $\lambda_p(ah_i) \in PM_p(G)$. Thus $a \cdot T = \lim_i a \cdot \lambda_p(h_i) = \lim_i \lambda_p(ah_i) \in PM_p(G)$. $\square$

**Remark 2.9.** We say that $G$ is *p-weakly amenable* whenever $A_p(G)$ – if we like $A_{p,c}(G)$ – admits a net $(\varphi_i)$ of elements which is bounded in $\mathcal{M}_p(G)$ and tends to 1 uniformly on compact sets. For $p = 2$, this is the property of weak amenability of de Cannière and Haagerup [7]. Remarks in the proof of Theorem 1.1, in Section 3 below, show that the assumption of 2-weak amenability is sufficient to obtain general $p$-weak amenability.

Suppose that $G$ is $p$-weakly amenable. Using the density of $N_c^p(G)$ in $N^p(G)$, it is easy to conclude that $\varphi_i \cdot \omega$ tends in norm to $\omega$, for any $\omega$ in $N^p(G)$. Hence, by the last corollary, any $T$ in $CV_p(G)$ is approximated weak* by the (bounded) net of elements $\varphi_i \cdot T$ from $PM_p(G)$. Hence we have proved Theorem 1.1 for such groups. For amenable groups, this is the proof of Herz [15]. For weakly amenable groups, this proof is hinted at by Cowling [3].

# 3 Proof of the approximation result, Theorem 1.1

We recall that $\mathcal{M}_2(G)$ is a dual space. We identify $L^1(G)$ with its evident image in $\mathcal{M}_2(G)^*$, acting by integration, and let $Q(G) = \overline{L^1(G)}^{\mathcal{M}_2(G)^*}$, the closure of $L^1(G)$ in $\mathcal{M}_2(G)^*$. De Cannière and Haagerup [7] show that $\mathcal{M}_2(G) \cong Q(G)^*$, and thus we define a weak*-topology on $\mathcal{M}_2(G)$.

As in Haagerup and Kraus [9], we say that $G$ admits the *approximation property* if $1 \in \overline{A_p(G)}^{w*}$, i.e. there is a net $(\varphi_i) \subset A_2(G)$ – we may suppose, in fact that $(\varphi_i) \subset A_{2,c}(G)$ – for which $\langle 1, q\rangle = \lim_i \langle \varphi_i, q\rangle$ for $q$ in $Q(G)$.

The following claim is implicit in [9] (see notes prior to Proposition 1.3, there), but the proof offered seems incomplete since it is not known if translations on $\mathcal{M}_2(G)$ are continuous in the translating variable.

**Lemma 3.1.** *If $f \in L^1(G)$ and $\varphi \in \mathcal{M}_2(G)$, then $f * \varphi \in \mathcal{M}_2(G)$.*

*Proof.* Since we have a contractive embedding $\mathcal{M}_2(G) \hookrightarrow L^\infty(G)$, we have that for $g$ in $L^1(G)$ that $\|g\|_Q \leq \|g\|_1$. Further $\mathcal{M}_2(G)$ is closed under left translations, and translation operators are isometries, so $Q(G)$ is a homogeneous space for left translation, and thus admits left convolution by elements of $L^1(G)$. Also, $\mathcal{M}_2(G)$ consists of weakly almost periodic functions (see the observation of Xu [24]) and hence of uniformly continuous functions, so $f*\varphi$ makes sense as a uniformly continuous function. Now we have

$$\begin{aligned}
&\sup\left\{\left|\int_G f*\varphi(s)g(s)\,ds\right| : g \in L^1(G),\ \|g\|_Q \leq 1\right\} \\
&\quad= \sup\left\{\left|\int_G \varphi(s)\tilde{f}*g(s)\,ds\right| : g \in L^1(G),\ \|g\|_Q \leq 1\right\} \\
&\quad\leq \|f\|_1 \sup\left\{\left|\int_G \varphi(s)g(s)\,ds\right| : g \in L^1(G),\ \|g\|_Q \leq 1\right\} = \|f\|_1\|\varphi\|_{\mathcal{M}_2}
\end{aligned}$$

where $\tilde{f}(t) = \Delta(t^{-1})f(t^{-1})$. Hence $f*\varphi \in \mathcal{M}_2(G)$. □

The fact that Hilbert spaces, which are exactly the [SQ$_2$]-spaces, are also [SQ$_p$] spaces (see [17]) gives a contractive embedding $\mathcal{M}_2(G) \hookrightarrow \mathcal{M}_p(G)$.

The following construction is modelled after Proposition 1.3 of [9].

**Lemma 3.2.** *Fix $a$ in $A_{p,c}(G)$ with $a \geq 0$ and $\int_G a = 1$. For $T$ in $CV_p(G)$ and $\omega$ in $N^p(G)$ let*

$$q_{T,\omega} \in \mathcal{M}_2(G)^* \text{ be given by } q_{T,\omega}(\varphi) = \langle T, (a*\varphi)\cdot\omega\rangle.$$

*Then $q_{T,\omega} \in Q(G)$.*

*Proof.* It is evident that $\|q_{T,\omega}\|_{\mathcal{M}_2^*} \leq \|T\|\|\omega\|_{N^p(G)}$. Hence by norm approximation, it suffices to show that if $\omega \in N^p_c(G)$, say $\omega \in N^p(K)$ for some $K$ in $\mathcal{K}(G)$, then $q_{T,\omega}(\varphi) = \int_G \varphi g$ for some $g$ in $L^1(G)$. We may further assume $K$ is so large that $a \in A_p(K)$. With these assumptions, let

$S = \mathrm{supp}(a)^{-1}\mathrm{supp}(P\omega)$, and we have

$$\begin{aligned}(a*\varphi)(s)P\omega(s) &= \int_G a(t)\varphi(t^{-1}s)P\omega(s)\,dt \\ &= \int_G a(t)1_S(t^{-1}s)\varphi(t^{-1}s)P\omega(s)\,dt \\ &= \int_G 1_S(t^{-1})\varphi(t^{-1})a_t(s)P\omega(s)\,dt.\end{aligned}$$

Let $L = KK^{-1}S^{-1} \cup KK^{-1}$ in $\mathcal{K}(G)$, which contains the support of each $a_t P\omega$. We note that $t \mapsto a_t : S^{-1} \to A_p(L)$ is continuous. Indeed, on elementary tensors $\xi \otimes f$ in $N^p(K)$, we have that $(\xi * \check{f})_t = \xi * (\lambda_p(t)f)^\vee$ and left translation is continuous on elements of $L^p(G)$. Hence we realize

$$(a*\varphi)P\omega = \int_G 1_S(t^{-1})\varphi(t^{-1})[a_t\,P\omega]\;dt$$

as a Bochner integral in $A_p(L)$, i.e. respecting the norm $\|\cdot\|_{A_p(L)}$. Then let

$$g(t) = 1_S(t)\Delta(t^{-1})\langle T, a_{t^{-1}}\,P\omega\rangle$$

which defines an element of $L^1(G)$. We may thus use the dual pairing (2.2) to compute

$$\begin{aligned}\int_G \varphi g &= \int_G 1_S(t^{-1})\varphi(t^{-1})\langle T, a_t\,P\omega\rangle\,dt \\ &= \left\langle T, \int_G 1_S(t^{-1})\varphi(t^{-1})[a_t\,P\omega]\right\rangle = \langle T, (a*\varphi)\cdot\omega\rangle.\end{aligned}$$

This establishes the desired claim. □

*Proof of Theorem 1.1.* Let $a$ be as in Lemma 3.2 and $(\varphi_i) \subset A_{2,c}(G)$ be a net converging weak* to 1. Note that $A_{2,c}(G) \subseteq \mathcal{M}_2(G) \subseteq \mathcal{M}_p(G)$, and hence $A_{2,c}(G)$ is a space of compactly supported elements of $\mathcal{M}_p(G)$. The argument just before Corollary 2.4 tells us that any compactly supported element of $\mathcal{M}_p(G)$ is in $A_{p,c}(G)$. Since $A_p(G)$ is a homogeneous space for left translations, we find that each $a * \varphi_i \in A_p(G)$.

Given $T$ in $CV_p(G)$, let $q_{T,\omega}$ be as in Lemma 3.2. Then we see that

$$\begin{aligned}\langle (a*\varphi_i)\cdot T, \omega\rangle &= \langle T, (a*\varphi_i)\cdot\omega\rangle = q_{T,\omega}(\varphi_i) \\ &\xrightarrow{i} q_{T,\omega}(1) = \langle T, (a*1)\cdot\omega\rangle = \langle T,\omega\rangle\end{aligned}$$

since $a * 1 = \left[\int_G a\right] 1 = 1$, by assumption on $a$. But Corollary 2.8 provides that each $(a*\varphi_i)\cdot T \in PM_p(T)$, and hence so too the weak* limit of this net, $T$, is also in $PM_p(G)$. □

Hints of the Theorem 1.1 are given by Cowling [3]. Given how heavily we have exploited his results, i.e. Theorem 2.2, we suspect he knew how to prove this.

**Remark 3.3.** The space $\mathcal{M}_p(G)$ generally has a predual $Q_p(G)$, as was verified by T. Miao in an unpublished manuscript. Using this, An, Lee and Ruan [1] defined the *p-approximation property* ($p$-AP), which is having 1 in the weak* closure of $A_p(G)$ in $\mathcal{M}_p(G)$. As verified by Vergara [23], $p$-AP is equivalent to $p'$-AP, and if $2 \leq p \leq q < \infty$, then $p$-AP implies $q$-AP. Hence $p$-AP for $p \neq 2$ is ostensibly more general than approximation property (2-AP). However, it is shown in [23] that certain canonical higher rank simple Lie groups fail $p$-AP for any $p$ – the same ones known to Lafforgue and de la Salle [18] and Haagerup and de Laat [11] – giving convincing evidence that, at least for connected groups and lattices within, that $p$-AP is equivalent to 2-AP. No examples are known of groups admitting $p$-AP for some $p$, but not 2-AP.

Nonetheless, it is simple to modify the Lemmas of this section to accomodate the assumption of $p$-AP. We refer the reader [23] for details.

# 4 Proof of the commutation results, Theorem 1.2

Theorem 2.2, above, provides a very useful approximation of a convoluter $T$ in $CV_p(G)$ by a bounded net of operators $(\lambda_p(h_i))$ where each $h_i \in CV_p^p(G)$. If we were conducting approximations for $p = 2$, then we need only consider self adjoint $T = T^*$ and we need not worry about determining the structure of $\lambda_p(h_i)^*$. For $p \neq 2$ we have no such luck. We overcome this difficulty below. Furthermore, in the spirit of Theorem 2.2, we give a careful pointwise approximation of a convoluter, on a dense subspace, by convolutions with elements of $L^1(G)$.

We note that $f \mapsto \tilde{f}$, where $\tilde{f}(t) = \Delta(t^{-1})f(t^{-1})$ for (locally) a.e. $s$, defines an isometric involution on $L^1(G)$, and clearly defines a linear map on $L^1_{\mathrm{loc}}(G)$ which satisfies $(h * f)^{\sim} = \tilde{f} * \tilde{h}$ for any convolvable pair of elements. Hence if $(f_i)$ is a contractive approximate identity on $L^1(G)$, then so too is $(\tilde{f}_i)$. Furthermore, if $f \in L^1(G)$, then $\lambda_p(f)^* = \lambda_{p'}(\tilde{f})$, as is standard to check. We also observe that since $\rho_p(s)^* = \rho_{p'}(s^{-1})$ for all $s$ in $G$, we have $\{T^* : T \in CV_p(G)\} = CV_{p'}(G)$.

**Theorem 4.1.** *Let $T \in CV_p(G)$.*

(i) *Then there is a net of elements* $(k_i)$ *in* $CV_p^p(G)$ *for which* $(\lambda_p(k_i))$ *is a bounded net in* $CV_p(G)$ *converging strong operator to* $T$*, and also* $(\tilde{k}_i)$ *is a net in* $CV_{p'}^{p'}(G)$ *for which* $(\lambda_{p'}(\tilde{k}_i))$ *is bounded in* $CV_{p'}(G)$ *and converges to* $T^*$.

(ii) *Given* $g, f$ *in* $C_c(G)$ *and* $\varepsilon > 0$*, there is* $k$ *in* $C_c(G)$ *for which*

$$\|Tf - \lambda_p(k)f\|_p < \varepsilon \text{ and } \|T^*g - \lambda_{p'}(\tilde{k})g\|_{p'} < \varepsilon.$$

*Here* $k$ *is dependant upon the choices of* $g, f$ *and* $\varepsilon$.

*Proof.* (i) As in Theorem 2.2 we let $(f_i) \subset C_c(G)$ be a contractive approximate identity for $L^1(G)$ and set $h_i = Tf_i$. We note that

$$\lambda_p(h_i) = \lambda_p(Tf_i) = T\lambda_p(f_i).$$

Let $g, f \in C_c(G)$. Since $\tilde{h}_i \in L^1_{\mathrm{loc}}(G)$, $\tilde{h}_i * g \in L^1_{\mathrm{loc}}(G)$ and, letting $S = \mathrm{supp} f \cup \mathrm{supp} g$, we have

$$\begin{aligned}
\int_G \tilde{h}_i * g(s) f(s)\, ds &= \int_S \int_{SS^{-1}} \Delta(t^{-1}) h_i(t^{-1}) g(t^{-1}s) f(s)\, dt\, ds \\
&= \int_{SS^{-1}} \int_S g(s) h_i(t) f(t^{-1}s)\, ds\, dt \\
&= \langle g, \lambda_p(h_i) f\rangle = \langle g, T\lambda_p(f_i) f\rangle = \langle \lambda_{p'}(\tilde{f}_i) T^* g, f\rangle.
\end{aligned}$$

Hence by density it follows that $\tilde{h}_i * g = \lambda_{p'}(\tilde{f}_i)T^* g$, and hence $\tilde{h}_i \in CV_{p'}^1(G)$ with

$$\lambda_p(\tilde{h}_i) = \lambda_{p'}(\tilde{f}_i)T^*.$$

We do not know of a means to see that $\tilde{h}_i \in CV_{p'}^{p'}(G)$. We use another convolution to overcome this difficulty.

Let $k_i = f_i * h_i$. It is clear that $k_i \in CV_p^p(G)$ with $\lambda_p(k_i) = \lambda_p(f_i)\lambda_p(h_i)$, and hence $\|\lambda_{p'}(\tilde{k}_i)\| \le \|f_i\|_1 \|\lambda_p(h_i)\|$. Likewise $\tilde{k}_i = \tilde{h}_i * \tilde{f}_i \in CV_{p'}^1(G)$ with $\lambda_{p'}(\tilde{k}_i) = \lambda_{p'}(\tilde{h}_i)\lambda_p(\tilde{f}_i)$, with $\|\lambda_{p'}(\tilde{k}_i)\| \le \|\lambda_{p'}(\tilde{h}_i)\| \|f_1\|_1$. Moreover, as $T^* \in CV_{p'}(G)$ we have that $T^*(\tilde{f}_i) \in CV_{p'}^{p'}(G)$ and

$$\lambda_{p'}(\tilde{k}_i) = \lambda_{p'}(\tilde{h}_i)\lambda_p(\tilde{f}_i) = \lambda_{p'}(\tilde{f}_i)T^*\lambda_p(\tilde{f}_i) = \lambda_{p'}(\tilde{f}_i * T^*\tilde{f}_i)$$

so $\tilde{k}_i = \tilde{f}_i * T^*(\tilde{f}_i) \in CV_{p'}^{p'}(G)$ too. We have that

$$\begin{aligned}
\|\lambda_p(k_i)f - Tf\|_p &= \|f_i * [Tf_i * f] - Tf\|_p \\
&\le \|f_i * [Tf_i * f] - f_i * Tf\|_p + \|f_i * Tf - Tf\|_p \\
&\le \|f_i\|_1 \|Tf_i * f - Tf\|_p + \|f_i * Tf - Tf\|_p
\end{aligned}$$

which tends to zero. By uniform boundedness of the operators $\lambda_p(k_i)$ and density of $C_c(G)$ in $L^p(G)$, we see that $\lambda_p(k_i)$ tends to $T$ in the strong operator topology. Similarly, the net $(\lambda_{p'}(\tilde{k}_i))$ is bounded and tends to $T^*$ in the strong operator topology.

(ii) Let $\delta > 0$. Let $i$ be such that

$$\|Tf - k_i * f\|_p < \delta \text{ and } \|T^*g - \tilde{k}_i * g\|_{p'} < \delta.$$

Inner regularity of the Haar measure provides $K$ in $\mathcal{K}(G)$ for which

$$\|k_i - 1_K k_i\|_p < \delta \text{ and } \|\tilde{k}_i - 1_{K^{-1}}\tilde{k}_i\|_{p'} < \delta.$$

Let $k = 1_K k_i$ which is in $L^1 \cap L^p(G)$, and note that $\tilde{k} = 1_{K^{-1}}\tilde{k}_i$. Since $l * f = \rho_p(\Delta^{-1/p}\check{f})l$ for $l$ in $L^p(G)$, we have that

$$\begin{aligned}\|Tf - k * f\|_p &\leq \|Tf - k_i * f\|_p + \|k_i * f - k * f\|_p \\ &< \delta + \|\rho_p(\Delta^{-1/p}\check{f})\|\delta.\end{aligned}$$

Likewise we have that

$$\|T^*g - \tilde{k} * g\|_{p'} < \delta + \|\rho_{p'}(\Delta^{-1/p'}\check{g})\|\delta.$$

Hence an obvious choice of $\delta$ yields the desired inequalities. $\square$

We note that the sorts of global estimates found in (i), above, were required to obtain the pointwise estimates of (ii).

**Remark 4.2.** As noted in the introduction, there is a self-inverting isometry $U$ on $L^p(G)$, which intertwines $\lambda_p$ and $\rho_p$: $U\lambda_p(s) = \rho_p(s)U$ for $s$ in $G$. Hence $U^*$ intertwines $\lambda_{p'}$ and $\rho_{p'}$. We let $CV_p'(G) = \lambda_p(G)'$ so

$$CV_p'(G) = (U\rho_p(G)U)' = U\rho_p(G)'U = U\, CV_p(G)\, U.$$

If $S \in CV_p'(G)$ and $g, f \in C_c(G)$, then $U^*g, Uf \in C_c(G)$. Hence, given $\varepsilon > 0$ Theorem 4.1 (ii) provides $k'$ in $L^1(G)$ for which

$$\|USUUf - \lambda_p(k')Uf\|_p < \varepsilon \text{ and } \|(USU)^*U^*g - \lambda_{p'}(\tilde{k}')U^*g\|_{p'} < \varepsilon$$

hence multiplying the arguments of the expressions by the respective isometries $U$ and $U^*$ we obtain

$$\|Sf - \rho_p(k')f\|_p < \varepsilon \text{ and } \|S^*g - \rho_{p'}(\tilde{k}')g\|_{p'} < \varepsilon.$$

The key reason we have Theorem 4.1, is that for $k, k'$ in $L^1(G)$ we have $\lambda_p(k)\rho_p(k') = \rho_p(k')\lambda_p(k)$. We do not know how to verify directly that $\lambda_p(h)\rho_p(h') = \rho_p(h')\lambda_p(h)$ for $h, h'$ in $CV_p^p(G)$.

*Proof of Theorem 1.2.* Let $PM_p'(G) = U\, PM_p(G)\, U = \overline{\mathrm{lin}}^{w*}\rho_p(G)$. We have that

$$PM_p(G)' = \lambda_p(G)' = CV_p'(G) \supseteq PM_p'(G) \text{ and } PM_p'(G)' = CV_p(G).$$

Thus it follows that

$$PM_p(G)'' = CV_p'(G)' \subseteq PM_p'(G)' = CV_p(G).$$

Hence we will be done once we show that $CV_p(G) \subseteq CV_p'(G)'$, i.e. if $T \in CV_p(G)$ and $S \in CV_p'(G)$, we wish to see that $TS = ST$.

To this end, fix $g, f$ in $C_c(G)$. As in Theorem 4.1 (ii) and Remark 4.2, given $\varepsilon > 0$, find $k, k'$ in $L^1(G)$ for which

$$\begin{aligned} &\|Tf - \lambda_p(k)f\|_p < \varepsilon \text{ and } \|T^*g - \lambda_{p'}(\tilde{k})g\|_{p'} < \varepsilon \\ &\|Sf - \rho_p(k')f\|_p < \varepsilon \text{ and } \|S^*g - \rho_{p'}(\tilde{k'})g\|_{p'} < \varepsilon. \end{aligned}$$

Then

$$\begin{aligned} &|\langle g, TSf\rangle - \langle g, \lambda_p(k)\rho_p(k')f\rangle| \\ &\quad \le |\langle T^*g, Sf - \rho_p(k')f\rangle| + |\langle T^*g - \lambda_{p'}(\tilde{k})g, \rho_p(k')f\rangle| \\ &\quad \le \|T\|\|g\|_{p'}\varepsilon + \varepsilon\|\rho_p(k')f\|_p \le (\|T\|\|g\|_{p'} + \|S\|\|f\|_p + \varepsilon)\varepsilon \end{aligned}$$

and, by a similar calculation, we see that

$$|\langle g, STf\rangle - \langle g, \rho_p(k')\lambda_p(k)f\rangle| \le (\|S\|\|g\|_{p'} + \|T\|\|g\|_p + \varepsilon)\varepsilon.$$

Since $\lambda_p(k)\rho_p(k') = \rho_p(k')\lambda_p(k)$ for any $k, k'$ in $L^1(G)$, we can take $\varepsilon$ to 0 and see that

$$\langle g, TSf\rangle = \langle g, STf\rangle.$$

Hence by density we see that $\langle \xi, TSf\rangle = \langle \xi, STf\rangle$ for any $\xi$ in $L^{p'}(G)$ and $f$ in $L^p(G)$, so $TS = ST$, as desired. □

We obtain an "elementary" proof of Dixmier's result [8], i.e. without use of left Hilbert algebras. The common algebra in the result below is usually denoted $VN(G)$ and called the *group von Neuman algebra*.

**Corollary 4.3.** *We have that*

$$CV_2(G) = CV_2'(G)' = PM_2(G) = PM_2'(G)'.$$

*Proof.* Here $\lambda_2(G)$ is a self-adjoint subset of $\mathcal{B}(L^2(G))$, so $VN(G) = PM_2(G)$ is self-adjoint. The bicommutant theorem of von Neumann, [22, II.3.9] for example, tells us that $PM_2(G)''$ is the $\sigma$-strong$^*$ closure of $PM_2(G)$. As a convex subspace of $\mathcal{B}(L^2(G))$ is weak$^*$-closed if and only if it is $\sigma$-strong$^*$ closed (see [22, II.2.6]), we see that $PM_2(G)'' = PM_2(G)$. The rest follows from the theorem above. □

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