On convoluters on $L^p$-spaces

Daws, Matthew and Spronk, Nico

Available at http://clok.uclan.ac.uk/21777/


It is advisable to refer to the publisher’s version if you intend to cite from the work. http://dx.doi.org/10.4064/sm170601-24-11

For more information about UCLan’s research in this area go to http://www.uclan.ac.uk/researchgroups/ and search for <name of research Group>.

For information about Research generally at UCLan please go to http://www.uclan.ac.uk/research/

All outputs in CLoK are protected by Intellectual Property Rights law, including Copyright law. Copyright, IPR and Moral Rights for the works on this site are retained by the individual authors and/or other copyright owners. Terms and conditions for use of this material are defined in the http://clok.uclan.ac.uk/policies/
On convoluters on $L^p$-spaces

Matthew Daws
Leeds, United Kingdom
E-mail: matt.daws@cantab.net

Nico Spronk
Department of Pure Mathematics
University of Waterloo
Waterloo, Ontario, N2L3G1, Canada
E-mail: nspronk@uwaterloo.ca

November 29, 2017

Abstract

We prove two theorems about convolution operators on $L^p(G)$ for a locally compact group $G$. First, if $G$ has the approximation property, then the algebra of convoluters is the algebra of pseudo-measures. Second, the bicommutant of the algebra of pseudo-measures is the algebra of convoluters.

1 Introduction

Let $G$ be a locally compact group and $L^p(G)$ denote the $L^p$-space with respect to left Haar measure, for $p \in [1, \infty]$. The algebra of bounded operators $\mathcal{B}(L^p(G))$ for $p \in (1, \infty)$ admits a natural predual (which we specify in Section 2 below) and hence admits a specified weak* topology which is finer than the weak operator topology. We consider the two subalgebras, the pseudo-measures and convoluters, which are given by

$$PM_p(G) = \overline{\text{lin}}^* \lambda_p(G)$$ (weak* closed linear span)

$$CV_p(G) = \rho_p(G)'$$ (commutant)

2010 Mathematics Subject Classification: Primary 43A15; Secondary 22D12, 47L10.
Key words and phrases: convolter, pseudo-measure, approximation property.
where the right and left regular representations \( \lambda_p, \rho_p : G \to \mathcal{B}(L^p(G)) \) are given by

\[
\lambda_p(s)f(t) = f(s^{-1}t), \quad \rho_p(s)f(t) = \Delta(s)^{1/p}f(ts)
\]

for all \( s \) and a.e. \( t \). Here \( \Delta \) is the Haar modular function. Since \( \lambda_p(G) \subseteq \rho_p(G)' \) and commutant algebras are weak operator closed, hence weak* closed, we have that \( \mathcal{P}M_p(G) \subseteq \mathcal{C}V_p(G) \). Notice, moreover, that the invertible isometry \( U \) in \( \mathcal{B}(L^p(G)) \), given by \( Uf(t) = \Delta^{1/p}(t^{-1})f(t^{-1}) \), is self-inverse and intertwines \( \lambda_p \) and \( \rho_p \), which shows we may interchange the roles of left and right.

The approximation property was defined by Haagerup and Kraus [9]. A detailed definition is provided in Section 3 below.

**Theorem 1.1.** If \( G \) has the approximation property, then

\[
\mathcal{C}V_p(G) = \mathcal{P}M_p(G).
\]

A complete description of connected groups with the approximation property is provided by Haagerup, Knudby and de Laat [10]: any simple Lie quotient must have real rank 1. This builds on an enormous body of work, including [18, 11, 12]. The approximation property passes to closed subgroups, extension groups, and is passed up from lattices [9]. In particular, countably generated free groups enjoy this property, even the stronger property of weak amenability, defined by de Cannière and Haggerup [7].

The approximation property does not pass to general quotients: the finitely generated lattice \( \text{SL}_3(\mathbb{Z}) \) in \( \text{SL}_3(\mathbb{R}) \) is a quotient of some free group \( \mathbb{F}_n \). In fact, we have the following implications of properties of \( G \):

\[
\text{amenable} \Rightarrow \text{weakly amenable} \Rightarrow \text{approximation property}.
\]

The first implication is given by Leptin [19], i.e. a bounded approximate identity in the Fourier algebra \( A_2(G) \) satisfies the definition of weak amenability of [7]. For amenable \( G \), Theorem 1.1 is proved by Herz [15]; see Remark 2.9.

A certain \( p \)-approximation property, which is implied by the approximation property was defined by An, Lee and Ruan [1], and studied vigorously by Vergara [23]. Using this, Vergara strengthens our Theorem 1.1, but relies heavily on our methods. See Remark 3.3 below.

On a related note, we also give an elementary proof of the following. Recall that the intertwiner \( U \) is given above
Theorem 1.2. We have commutation results:

\[ PM_p(G)' = U CV_p(G)U \quad \text{and} \quad (U CV_p(G)U)' = CV_p(G). \]

In particular, we have bicommutant \( PM_p(G)'' = CV_p(G) \).

For \( p = 2 \) the commutation result of convoluters was proved by Dixmier [8] using left Hilbert algebras. Recently, Pham [21] gave an elementary proof in the \( p = 2 \) case. This proof is similar to the one offered here, but we wish to note that ours was conducted independently and posted on arXiv in 2016.

The present article is an updated version of our work [6], the main body of which was first posted to the arXiv in 2013. Both theorems appear to be folklore — see Cowling [3], and Herz [15] — but we have been unable to track down complete proofs. Theorem 1.1 was used in recent work of Öztöp and the second named author [20] to obtain, for a pair of groups \( G \) and \( H \) with approximation property, a tensor product description of \( A_p(G \times H) \). Given recent developments related to this work and interest in it, in particular [21, 23], we have elected to update our results and submit them for publication.

2 On preduals of pseudo-measures and convoluters

The entire goal of the present section is to present background for the proofs of our two main theorems. We also redevelop some methods of Cowling [3], who gave an innovative and insightful description of a predual of \( CV_p(G) \). The present methods give a new perspective and help to illuminate the role of the Herz-Schur multipliers which we discuss in 2.2 below.

We let \( p' \) be the conjugate index to \( p \), so \( \frac{1}{p} + \frac{1}{p'} = 1 \). We recall that \( \mathcal{B}(L^p(G)) \) is the dual of the space \( N^p(G) = L^{p'}(G) \otimes L^p(G) \) by way of dual pairing \( \langle T, \xi \otimes f \rangle = \langle \xi, Tf \rangle = \int_G \xi(t)[Tf](t) \, dt \). Elements \( \omega = \sum_{n=1}^{\infty} \xi_n \otimes f_n \) of \( N^p(G) \) may be viewed as functions \( \omega(s, t) = \sum_{n=1}^{\infty} \xi_n(s)f_n(t) \) up to marginal almost everywhere (m.a.e.) equivalence; i.e. excepting marginally null sets, \((N_1 \times G) \cup (G \times N_2)\), where \( N_1 \) and \( N_2 \) are Haar null sets.

We let \( P : N^p(G) \to A_p(G) \subseteq C_0(G) \) be given for \( s \) in \( G \) by

\[
P \left( \sum_{n=1}^{\infty} \xi_n \otimes f_n \right)(s) = \sum_{n=1}^{\infty} \langle \xi_n, \lambda_p(s)f_n \rangle = \sum_{n=1}^{\infty} \xi_n * \tilde{f}_n
\]

where \( \tilde{f}(t) = f(t^{-1}) \) a.e., i.e. \( P \omega(s) = \int_G \omega(t, s^{-1}t) \, dt \). Hence \( A_p(G) \) is normed as the coimage of \( P \), i.e. \( A_p(G) \cong N^p(G)/\ker P \) isometrically. It
is straightforward to see that $PM_p(G)$ and $CV_p(G)$ admit the following pre-annihilators in $N^p(G)$:

$$\begin{align*}
\perp PM_p(G) & = \ker P \\
\perp CV_p(G) & = \lim \{ \rho_p(t^{-1}) \xi \otimes f - \xi \otimes \rho_p(t)f : \\
& t \in G, \xi \in L^p(G), f \in L^p(G) \}
\end{align*}$$

where we note that adjoint of these right translation operators is given by $\rho_p(t)^* = \rho_{p'}(t^{-1})$.

It is standard that $PM_p(G) \cong A_p(G)^*$ and $CV_p(G) \cong [N^p(G)/\perp CV_p(G)]^*$ which gives us distinguished preduals of these algebras of operators.

### 2.1 $N^p(G)$ as an algebra

Herz [13] showed that $A_p(G)$ is always a subalgebra of $C_0(G)$, and further showed ([14]) that it has Gelfand spectrum given by evaluations at points of $G$. We note that $A_p(G)$ is known as the Figà-Talamanca–Herz algebra. Herz’s technique for showing that $A_p(G)$ is an algebra lifts naturally to $N^p(G)$. We note that the amplification $T \mapsto T \otimes I : \mathcal{B}(L^p(G)) \to \mathcal{B}(L^p(G \times G)) \cong \mathcal{B}(L^p(G \times G))$ is an isometry. We consider the fundamental isometry $W_p$ in $L^p(G \times G)$ given for a.e. $(s, t)$ by

$$W_p f(s, t) = f(s, st).$$

We note that $W_p$ is invertible with $W_p^* = W_p^{-1}$. We define a co-product on $\mathcal{B}(L^p(G))$, $\Gamma : \mathcal{B}(L^p(G)) \to \mathcal{B}(L^p(G \times G))$ by

$$\Gamma(T) = W_p^{-1}(T \otimes I)W_p.$$ 

This spatially implemented map is evidently weak*-weak* continuous, and hence admits a preadjoint. We note that $N^p(G) \otimes N^p(G)$ comprises a dense subspace of $N^p(G \times G)$ given on elementary tensors of elementary tensors (“really elementary tensors”) by $(\xi \otimes f) \otimes (\eta \otimes g) \mapsto (\xi \otimes \eta) \otimes (f \otimes g)$. Let us compute $\Gamma_*$ on these really elementary tensors. We have

$$\begin{align*}
\langle T, \Gamma_*( (\xi \otimes f) \otimes (\eta \otimes g) ) \rangle & = \langle W_p^{-1}(T \otimes I)W_p, (\xi \otimes f) \otimes (\eta \otimes g) \rangle \\
& = \langle W_p(\xi \otimes \eta), (T \otimes I)W_p(f \otimes g) \rangle \\
& = \int_G \int_G \xi(s)\eta(st)(T \otimes I)[(s, t) \mapsto f(s)g(st)] ds dt \\
& = \int_G \langle \xi \eta_t, T(fg_t) \rangle dt
\end{align*}$$
On convoluters

where \( \eta_t(s) = \eta(st) \), for example. Hence we see that

\[
(2.1) \quad \Gamma_*(((\xi \otimes f) \otimes (\eta \otimes g))) = \int_G (\xi \eta_t) \otimes (fg_t) \, dt.
\]

The equation

\[
\int_G \int_G (\xi \eta_t \partial_s) \otimes (fg_t h_s) \, dt \, ds = \int_G \int_G (\xi \eta_t \partial_{ts}) \otimes (fg_t h_{ts}) \, ds \, dt
\]

demonstrates that \( \Gamma_* \) is an associative product. We note that though \( \Gamma(\lambda_p(r)) = \lambda_p(r) \otimes \lambda_p(r) \), which implies that \( \Gamma|_{PM_p(G)} \) is cocommutative, we have for a multiplication operator \( M_\varphi \), where \( \varphi \in L^\infty(G) \), that \( \Gamma(\lambda_p) = \lambda_p \otimes \lambda_p \), which shows that \( \Gamma \) is not generally cocommutative. Hence \( \Gamma_* \) is not commutative.

Let \( K(G) \) denote the family of all compact localisations of the identity in \( G \). For \( K \) in \( K(G) \) we let \( L^p(K) = 1_K L^p(G) \), which is a 1-complemented subspace. Likewise we define \( L^p(K) \) and

\[
\mathcal{N}^p(K) = L^p(K) \hat{\otimes} L^p(K)
\]

which is a subspace of \( N^p(G) \).

**Proposition 2.1.** The space \( \mathcal{M}^p(G) = \ker P \) is a \( \Gamma_* \)-ideal in \( N^p(G) \), while each of \( \mathcal{C}^p(G) \) and \( \mathcal{N}^p(K) \), for \( K \) in \( K(G) \), are right ideals.

**Proof.** That \( P : N^p(G) \to C_0(G) \) is \( \Gamma_* \)-pointwise multiplicative is shown in [13], hence \( \mathcal{M}^p(G) = \ker P \) is an ideal. Let us consider the case of \( \mathcal{C}^p(G) \), manually. On really elementary tensors we have

\[
\Gamma_*(((\xi \otimes f) \otimes (\eta \otimes g)) = \int_G (\xi \eta_t \otimes (\eta \otimes g)) = \Gamma_*(((\xi \otimes f) \otimes (\eta \otimes g))
\]

which is an integral of elements in \( \mathcal{C}^p(G) \). That each \( \mathcal{N}^p(K) \) is a right ideal follows readily from (2.1). Indeed if \( \xi \otimes f \in \mathcal{N}^p(K) \) then

\[
(1_K \otimes 1_K)\Gamma_*(((\xi \otimes f) \otimes (\eta \otimes g)) = \int_G 1_K \xi \eta_t \otimes 1_K fg_t \, dt
\]

so \( \Gamma_*(((\xi \otimes f) \otimes (\eta \otimes g)) \in \mathcal{N}^p(K) \). \( \square \)
It will be convenient below, to consider the following space of elements. We let $L^1_{\text{loc}}(G)$ denote the space of locally a.e. equivalence classes of measurable functions on $G$ which are integrable on any compact set. An application of Hölder’s inequality shows that $L^q(G) \subseteq L^1_{\text{loc}}(G)$ for any $q$ in $[1, \infty]$. If $h \in L^1_{\text{loc}}(G)$ and $g \in C_c(G)$ then $h \ast g$ is well-defined as an element of $L^1_{\text{loc}}(G)$. We let
\[
CV^1_p(G) = \{h \in L^1_{\text{loc}}(G) : \sup \{\|h \ast g\|_p : g \in C_c(G), \|g\|_p \leq 1\} < \infty\}.
\]
Each element $h$ of $CV^1_p(G)$ defines an operator $\lambda_p(h)$ in $B(L^p(G))$. By testing on elements $\rho_p(s^{-1})g \otimes f - g \otimes \rho_p(s)f$ where $f, g \in C_c(G)$, and applying a standard density argument, we see that $\lambda_p(h) \in CV^1_p(G)$.

The next result is really a repackaging of the main result of Cowling [3], and is used extensively through the rest of this note. The second part is a localization theorem.

**Theorem 2.2.**  
(i) Each $T$ in $CV^1_p(G)$ may be approximated in the strong operator topology by a bounded net of elements $\lambda_p(h_i)$ where each $h_i \in CV^1_p(G) = CV^1_p \cap L^p(G)$.

(ii) For each $K$ in $\mathcal{K}(G)$, we have that
\[
\dag PM_p(G) \cap N^p(K) = \dag CV^1_p(G) \cap N^p(K).
\]

**Proof.** (i) Let $T \in CV^1_p(G)$. Let $(f_i) \subset C_c(G)$ be a contractive approximate identity for $L^1(G)$. Then for $g \in C_c(G)$ we have
\[
T(f_i \ast g) = T(\rho_p(\Delta^{-1/p} \hat{g})f_i) = \rho_p(\Delta^{-1/p} \hat{g})T(f_i) = T(f_i) \ast g
\]
where $\hat{g}(t) = g(t^{-1})$ a.e. Notice then, that
\[
\|T(f_i) \ast g\|_p = \|T(f_i \ast g)\|_p \leq \|T\|\|f_i \ast g\|_p \leq \|T\|\|g\|_p
\]
and
\[
\|T(f_i) \ast g - Tg\|_p = \|T(f_i \ast g - g)\|_p \xrightarrow{i} 0.
\]
It follows that $h_i = T(f_i)$ gives the desired net of elements in $CV^1_p(G)$.

(ii) Since $\dag PM_p(G) \supseteq \dag CV^1_p(G)$, it suffices to show that $\dag PM_p(G) \cap N^p(K) \subseteq \dag CV^1_p(G)$. From (i), above, it suffices to show that for any $h$ in $CV^1_p(G)$, that $\lambda_p(h)$ annihilates $\dag PM_p(G) \cap N^p(K)$.

Now let $\xi \otimes f \in N^p(K)$, and let $(f_n)$ be a sequence in $C_c(G)$ whose supports are in the neighbourhood $KK^{-1}$ of $K$ and converges to $f$ in $L^p(G)$.
Then we have that
\[
\langle \xi, \lambda_p(h) f \rangle = \lim_n \langle \xi, \lambda_p(h) f_n \rangle = \lim_n \int_K \xi(t) h \ast f_n(t) \, dt
\]
\[
= \lim_n \int_K \xi(t) \int_{K^{2K-1}} h(s) f_n(s^{-1}t) \, ds \, dt
\]
\[
= \lim_n \int_{K^{2K-1}} h(s) \int_K \xi(t) f_n(s^{-1}t) \, dt \, ds
\]
\[
= \lim_n \int_{K^{2K-1}} h(s) P(\xi \otimes f_n)(s) \, ds
\]
\[
= \int_{K^{2K-1}} h(s) P(\xi \otimes f)(s) \, ds
\]
where each interchange of integrals is justified by Fubini’s theorem, as \((\xi \otimes h)1_{K^{2K-1}}\) is integrable and each \(f_n\) is bounded, and the limits are justified by \(L^p\)-convergence of \(f_n\), then uniform convergence of \(P(\xi \otimes f_n)\). Hence, if \(\omega = \sum_{n=1}^\infty \xi_n \otimes f_n \in \mathcal{N}(\mathcal{M}) \cap N^p(K)\), then uniform convergence of \(\sum_{n=1}^\infty P(\xi_n \otimes f_n) = P \omega = 0\) provides that
\[
\langle \lambda_p(h), \omega \rangle = \sum_{n=1}^\infty \langle \xi_n, \lambda_p(h) f_n \rangle = \sum_{n=1}^\infty \int_{K^{2K-1}} h(s) P(\xi_n \otimes f_n)(s) \, ds = 0
\]
which is the desired annihilation condition.

We note that proving Theorem 1.1 amounts, in effect, to showing that \(\lambda_p(CV_1^p(G)) \subseteq \mathcal{M}(G)\), even that \(\lambda_p(CV_1^p(G)) \subseteq \mathcal{M}(G)\). Attacking this directly seems very delicate; see Cowling’s approach [2] for certain simple connected Lie groups with real rank 1. We will proceed through a different sequence of observations.

**Corollary 2.3.** (i) The space \(\bigcup_{K \in \mathcal{K}(G)} \mathcal{N}(\mathcal{M}(G) \cap N^p(K)\) is dense in \(\mathcal{N}(G)\).

(ii) Every \(\Gamma^\ast\)-commutator is an element of \(\mathcal{N}(G)\). Hence \(\mathcal{N}(G)\) is a two-sided \(\Gamma^\ast\)-ideal.

**Proof.** (i) We observe that the family of elements
\[
\bigcup_{K \in \mathcal{K}(G)} \{ \rho_p(s^{-1})\xi \otimes f - \xi \otimes \rho_p(s) f : s \in K, \xi \in L'(K), f \in L^p(K) \}
\]
is contained in \(\bigcup_{K \in \mathcal{K}(G)} \mathcal{N}(\mathcal{M}(G) \cap N^p(K)\cup K^2),\) and its linear span is dense in \(\mathcal{N}(G)\).
(ii) Any commutator $\Gamma_\ast(\omega \otimes \mu - \mu \otimes \omega)$ can be approximated by elements of the form $\Gamma_\ast(\omega \otimes [1_K \otimes 1_K] - [1_K \otimes 1_K] \mu \otimes [1_K \otimes 1_K])$ where $K \in \mathcal{K}(G)$. Such elements are in $\perp PM_p(G) \cap N^p(K)$, hence in $\perp CV_p(G)$.

Now, if $\omega \in \perp CV_p(G)$ and $\mu \in N^p(G)$ we have

$$\Gamma_\ast(\mu \otimes \omega) = \Gamma_\ast(\omega \otimes \mu) + \Gamma_\ast(\mu \otimes \omega - \omega \otimes \mu) \in \perp CV_p(G)$$

which establishes the second fact. \hfill \Box

Hence $\Gamma_\ast$ induces a commutative product on

$$\overline{A}_p(G) = N^p(G)/\perp CV_p(G)$$

which is a predual of $CV_p(G)$. We let $\overline{P} : N^p(G) \to \overline{A}_p(G)$ denote the quotient map. We note that there is a further natural quotient map $\overline{P}(\omega) \mapsto P\omega : \overline{A}_p(G) \to A_p(G)$. It is shown by Cowling [3] that this is the Gelfand transform. In particular, semisimplicity of $\overline{A}_p(G)$ is equivalent to having $\perp CV_p(G) = \perp PM_p(G)$, i.e. $CV_p(G) = PM_p(G)$.

Now we let for $K$ in $\mathcal{K}(G)$

$$A_p(K) = P(N^p(K)).$$

We remark that if $K$ is an open subgroup of $G$, then our definition of $A_p(K)$ coincides with the usual definition. For $a$ in $A_p(K)$

$$\|a\|_{A_p(K)} = \inf\{\|\omega\|_{N^p(G)} : \omega \in N^p(K), a = P(\omega)\}$$

is a norm on $A_p(K)$.

The set $\bigcup_{K \in \mathcal{K}(G)} A_p(K)$ is the algebra $A_{p,c}(G)$ of compactly supported elements of $A_p(G)$. Indeed, if $u$ in $A_{p,c}(G)$ is supported on compact $S$, let $L \in \mathcal{K}(G)$ and $u_L = \frac{1}{m(L)}1_{SL} * \mathfrak{I}_L$ is 1 on $S$, hence $u = uu_L \in A_p(K)$ whenever $SL \cup L \subseteq K$.

The following will play a critical role in the proof of Lemma 3.2, and hence in the proof of Theorem 1.1.

**Corollary 2.4.** Let $N_p^c(G) = \bigcup_{K \in \mathcal{K}} N^p(K)$. Then $\overline{P}(N_p^c(G))$ is a dense ideal in $\overline{A}_p(G)$ which is algebraically isomorphic to $A_{p,c}(G)$. Furthermore, for $\omega$ in $N_p^c(G)$ we have

$$\|\overline{P}(\omega)\|_{\overline{A}_p(G)} = \inf\{\|P(\omega)\|_{A_p(K)} : \omega \in N^p(K)\}$$

and the pairing with $CV_p(G)$ depends only on $P(\omega)$; i.e. if $T \in CV_p(G)$ we may write

$$\langle T, \omega \rangle = \langle T, P\omega \rangle.$$
Proof. It is evident that $N^p(G)$ is a dense right $\Gamma_*$-ideal in $N^p(G)$, so $\overline{\mathcal{P}(N^p(G))}$ is a dense ideal in $A_p(G)$, by part (ii) of the prior corollary. We use part (i) of the prior corollary to see that

$$\left\| \overline{\mathcal{P}(\omega)} \right\|_{A_p(G)} = \text{dist}(\omega, \frac{1}{2}CV_p(G))$$

$$= \inf_{K \in \mathcal{K}(G)} \text{dist}(\omega, \frac{1}{2}PM_p(G) \cap N^p(K))$$

$$= \inf \{ \left\| P(\omega) \right\|_{A_p(K)} : K \in \mathcal{K}(G), \omega \in N^p(K) \}$$

where we note that the norm on $A_p(K)$ is given as the coimage: $A_p(K) \sim = N^p(K)/\ker P|_{N^p(K)}$, where $\ker P|_{N^p(K)} = \frac{1}{2}PM_p(G) \cap N^p(K)$. Hence if, further, $P\omega = 0$ then for $T$ in $CV_p(G)$ we have

$$|\langle T, \omega \rangle| \leq \left\| T \right\| \left\| \overline{\mathcal{P}(\omega)} \right\|_{A_p(G)} = 0$$

which establishes (2.2). \qed

We record the following localization principle. In terminology of Herz [15], (b), below, is the condition of being “formally of compact support”.

**Theorem 2.5.** The following are equivalent:

(a) $CV_p(G) = PM_p(G)$;

(b) for each $u$ in $A_{p,c}(G)$ we have

$$\left\| u \right\|_{A_p(G)} = \inf \{ \left\| u \right\|_{A_p(K)} : K \in \mathcal{K}(G), u \in A_p(K) \}; \text{ and}$$

(c) there is $C > 0$ such that for each $u$ in $A_{p,c}(G)$ we have

$$\inf \{ \left\| u \right\|_{A_p(K)} : K \in \mathcal{K}(G), u \in A_p(K) \} \leq C \left\| u \right\|_{A_p(G)}.$$

**Proof.** That (a) implies (b) is immediate form the last corollary, whilst that (b) implies (c) is obvious for $C \geq 1$. If (c) holds, then on the dense subspace $A_{p,c}(G)$, which we may identify with $\overline{\mathcal{P}(N^p(G))}$, we have that $\left\| \cdot \right\|_{A_p(G)}$ and $\left\| \cdot \right\|_{A_p(G)}$ are equivalent norms, so the map $\omega + \frac{1}{2}CV_p(G) \mapsto P(\omega)$ is injective, which implies that $\frac{1}{2}CV_p(G) = PM_p(G)$. The latter fact gives (a). \qed

### 2.2 The action of Herz-Schur multipliers

Let $[SQ_p]$ denote the class of spaces which are isometrically isomorphic to a subspace of a quotient of an $L^p$-space. Kwapien [17] noted that for $E$ in $[SQ_p]$ that the ampliation $T \mapsto T \otimes I_E : \mathcal{B}(L^p(G)) \to \mathcal{B}(L^p(G; E))$ is an
isometry. In fact, he showed that this property characterizes elements of the class $[SQ_p]$. See Dales et al [11] for an exposition on this. Hence, by duality, the map $\gamma_E : L^p(G; E^*) \otimes L^p(G; E) \to N^p(G)$ given on elementary tensors m.a.e. by

\begin{equation}
\gamma_E(\Xi \otimes F)(s, t) = \langle \Xi(s), F(t) \rangle_{E^*, E}
\end{equation}

is a quotient map, in particular a contraction. Also see Herz [14]. Let us define the $p$-Herz-Schur multiplier space by

$$\mathcal{M}_p(G) = \{ \varphi : G \to \mathbb{C} | \varphi(st^{-1}) = \langle \beta(s), \alpha(t) \rangle \text{ where each }$$

$$\alpha : G \to E, \beta : G \to E^* \text{ is continuous}$$

$$\text{ and bounded for some } E \in [SQ_p] \}$$

These were essentially described by Herz [15, 16], and the specific version we are using here was given by the first named author [5]. As shown in [5], $\mathcal{M}_p(G)$ is the algebra of “$p$-completely bounded multipliers” of $A_p(G)$, and a Banach algebra with respect to the norm

$$\|\varphi\|_{\mathcal{M}_p} = \inf \{ \|\beta\|_{\infty}\|\alpha\|_{\infty} : \varphi(st^{-1}) = \langle \beta(s), \alpha(t) \rangle \text{ as above} \}.$$ 

**Proposition 2.6.** (i) $N^p(G)$ is a Banach $\mathcal{M}_p(G)$-module via the action given m.a.e. by

$$\varphi \cdot \omega(s, t) = \varphi(st^{-1})\omega(s, t).$$

(ii) Each of $^1PM_p(G)$ and $N^p(K)$ ($K \in \mathcal{K}(G)$) are $\mathcal{M}_p(G)$-submodules, hence so too is $^1CV_p(G)$.

**Proof.** (i) This is an immediate application of the map $\gamma_E$ in (2.3), above. If $\varphi \in \mathcal{M}_p(G)$ with $\varphi(st^{-1}) = \langle \beta(s), \alpha(t) \rangle$, and $\xi \otimes f$ is an elementary tensor in $N^p(G)$, then let $\xi\beta(s) = \xi(s)\beta(s)$ in $L^p(G; E^*)$ and $f\alpha(t) = f(t)\alpha(t)$ in $L^p(G; E)$. Notice that $\|\xi\beta\|_{\gamma_p; E^*} \leq \|\xi\|_{\gamma_p}\|\beta\|_{\infty}$ and $\|f\alpha\|_{\gamma_p; E} \leq \|f\|_{\gamma_p}\|\alpha\|_{\infty}$. Then

$$\varphi \cdot (\xi \otimes f) = \gamma_E(\xi \beta \otimes f\alpha) \in N^p(G)$$

and hence

$$\|\varphi \cdot (\xi \otimes f)\|_{N^p} \leq \|\xi\beta\|_{\gamma_p; E^*}\|f\alpha\|_{\gamma_p; E} \leq \|\beta\|_{\infty}\|\alpha\|_{\infty}\|\xi\|_{\gamma_p}\|f\|_{\gamma_p}$$

from which it follows that $\|\varphi \cdot (\xi \otimes f)\|_{N^p} \leq \|\varphi\|_{\mathcal{M}_p}\|\xi \otimes f\|_{N^p}$.

(ii) If $\varphi \in \mathcal{M}_p(G)$ and $\omega \in \ker P = ^1PM_p(G)$ then

$$P(\varphi \cdot \omega)(s) = \int_G \varphi(t(s^{-1}t)^{-1})\omega(t, s^{-1}t) \, dt = \varphi(s)P\omega(s) = 0.$$
On convoluters

It is facile that each $N^p(K)$ is an $M_p(G)$-submodule. Thus each $\pm P M_p(G) \cap N^p(K)$ is an $M_p(G)$-submodule, and the result for $\pm CV_p(G)$ follows from the density result, Corollary 2.3 (i), above.

It is immediate that the action in (ii), above, induces actions of $M_p(G)$ on each of $A_p(G)$ and $A_p(K)$ (each by pointwise multiplication) and on $\overline{A_p(G)}$, making each a continuous module. The spaces $A_p(K)$ are shown to be Banach $A_p(G)$-modules in [20], by a method quite similar to that above. We note that the natural map $P(\omega) \mapsto P \omega : A_p(G) \to A_p(G)$ is an $M_p(G)$-module map.

We shall have need to consider the adjoint action on the dual space. That $CV_p(G)$ is a $M_p(G)$-module seems also to be shown by Herz [16].

**Corollary 2.7.** The algebras $CV_p(G)$ and $PM_p(G)$ are dual Banach $M_p(G)$-modules:

$$\langle \varphi \cdot T, \omega \rangle = \langle T, \varphi \cdot \omega \rangle.$$

If $\varphi \in M_p(G)$ and $h \in CV_p^1(G)$, then $\varphi h \in CV_p^1(G)$ and

$$\varphi \cdot \lambda_p(h) = \lambda_p(\varphi h).$$

**Proof.** The first statement being an evident adjoint module operation, we are left only to inspect the second statement. However, if $\xi \otimes f$ is an elementary tensor in $N_p^p(G)$, i.e. in $N^p(K)$ for some $K$, then arguments just as in the proof of Theorem 2.2 (ii) provide that

$$\langle \varphi \cdot \lambda_p(h), \xi \otimes f \rangle = \langle \lambda_p(h), \varphi \cdot (\xi \otimes f) \rangle = \int_{KK} h(s)P(\varphi \cdot (\xi \otimes f))(s) \, ds$$

$$= \int_{KK} \varphi(s)h(s) \int_K \xi(t)f(s^{-1}t) \, dt \, ds$$

$$= \langle \xi, (\varphi h) \ast f \rangle$$

where the last interchange of integrals is justified as

$$(s, t) \mapsto |\varphi(s)h(s)\xi(t)f(s^{-1}t)|$$

is integrable thanks to Hölder’s inequality and Tonelli’s theorem. If $f \in C_c(G)$ then

$$\| (\varphi h) \ast f \|_p = \sup \{ |\langle \varphi \cdot \lambda_p(h), \xi \otimes f \rangle | : \xi \in L^p(K), K \in K(G), \| \xi \|_p \leq 1 \}$$

$$\leq \| \varphi \cdot \lambda_p(h) \| \| f \|_p.$$ 

Then taking supremum over such $f$ with $\| f \|_p \leq 1$, reveals that $\varphi h \in CV_p^1(G)$, and further gives the desired formula. \[\square\]
We note that \( A_p(G) \subseteq \mathcal{M}_p(G) \), is an ideal within the latter space, and the inclusion is a contractive map. Indeed if \( a = \sum_{n=1}^{\infty} (\xi_n, \lambda_p(\cdot) f_n) \) in \( A_p(G) \), with \( \sum_{n=1}^{\infty} \|\xi_n\|_{p'} \|f_n\|_p < \infty \) and each \( \|\xi_n\|_{p'} \|f_n\|_p > 0 \), let \( \xi'_n = \|\xi_n\|^{1/p'} \|f_n\|^{1/p} \xi_n \) and \( f'_n = \|\xi_n\|^{1/p'} \|f_n\|^{1/p} \). Then \( \beta(s) = (\lambda_{p'}(s^{-1}) \xi'_n)_{n=1}^{\infty} \) in \( \ell^{p'}(\mathbb{N}, L^{p'}(G)) \) and \( \alpha(t) = (\lambda_p(t^{-1}) f'_n)_{n=1}^{\infty} \) in \( \ell^p(\mathbb{N}, L^p(G)) \) show that \( a \in \mathcal{M}_p(G) \), with \( \|a\|_{\mathcal{M}_p} \leq \|a\|_{A_p(G)} \).

Hence we get the following crucial result which will help us in the proof of Theorem 1.1.

Corollary 2.8. Let \( T \in CV_p(G) \) and \( a \in A_p(G) \). Then \( a \cdot T \in PM_p(G) \).

Proof. By a norm estimate, we may assume that \( a \in A_{p,c}(G) \). Furthermore, Theorem 2.2 (i) provides that \( T \) is a weak* limit of \( \lambda_p(h_i) \) for some \( h_i \) in \( CV^1_p(G) \). Then the prior corollary provides that each \( a \cdot \lambda_p(h_i) = \lambda_p(ah_i) \), where \( ah_i \in L^1_{loc}(G) \) with compact support, hence \( ah_i \in L^1(G) \), so \( \lambda_p(ah_i) \in PM_p(G) \). Thus \( a \cdot T = \lim_i a \cdot \lambda_p(h_i) = \lim_i \lambda_p(ah_i) \in PM_p(G) \).

Remark 2.9. We say that \( G \) is \( p \)-weakly amenable whenever \( A_p(G) \) – if we like \( A_{p,c}(G) \) – admits a net \( (\varphi_i) \) of elements which is bounded in \( \mathcal{M}_p(G) \) and tends to 1 uniformly on compact sets. For \( p = 2 \), this is the property of weak amenability of de Cannièere and Haagerup [7]. Remarks in the proof of Theorem 1.1 in Section 3 below, show that the assumption of 2-weak amenability is sufficient to obtain general \( p \)-weak amenability.

Suppose that \( G \) is \( p \)-weakly amenable. Using the density of \( N^p_v(G) \) in \( N^p(G) \), it is easy to conclude that \( \varphi_i \cdot \omega \) tends in norm to \( \omega \), for any \( \omega \) in \( N^p(G) \). Hence, by the last corollary, any \( T \) in \( CV_p(G) \) is approximated weak* by the (bounded) net of elements \( \varphi_i \cdot T \) from \( PM_p(G) \). Hence we have proved Theorem 1.1 for such groups. For amenable groups, this is the proof of Herz [15]. For weakly amenable groups, this proof is hinted at by Cowling [3].

3 Proof of the approximation result, Theorem 1.1

We recall that \( \mathcal{M}_2(G) \) is a dual space. We identify \( L^1(G) \) with its evident image in \( \mathcal{M}_2(G)^* \), acting by integration, and let \( Q(G) = \overline{L^1(G)^{\mathcal{M}_2(G)^*}} \), the closure of \( L^1(G) \) in \( \mathcal{M}_2(G)^* \). De Cannièere and Haagerup [7] show that \( \mathcal{M}_2(G) \cong Q(G)^* \), and thus we define a weak*-topology on \( \mathcal{M}_2(G) \).
As in Haagerup and Kraus [9], we say that $G$ admits the approximation property if $1 \in \overline{A_p(G)^{w^*}}$, i.e. there is a net $(\varphi_i) \subset A_2(G)$ – we may suppose, in fact that $(\varphi_i) \subset A_{2,c}(G)$ – for which $(1, q) = \lim_i \langle \varphi_i, q \rangle$ for $q$ in $Q(G)$.

The following claim is implicit in [9] (see notes prior to Proposition 1.3, there), but the proof offered seems incomplete since it is not known if translations on $M_2(G)$ are continuous in the translating variable.

**Lemma 3.1.** If $f \in L^1(G)$ and $\varphi \in M_2(G)$, then $f * \varphi \in M_2(G)$.

**Proof.** Since we have a contractive embedding $M_2(G) \hookrightarrow L^\infty(G)$, we have that for $g$ in $L^1(G)$ that $\|g\|_Q \leq \|g\|_1$. Further $M_2(G)$ is closed under left translations, and translation operators are isometries, so $Q(G)$ is a homogeneous space for left translation, and thus admits left convolution by elements of $L^1(G)$. Also, $M_2(G)$ consists of weakly almost periodic functions (see the observation of Xu [24]) and hence of uniformly continuous functions, so $f * \varphi$ makes sense as a uniformly continuous function. Now we have

$$
\sup \left\{ \left| \int_G f * \varphi(s)g(s) \, ds \right| : g \in L^1(G), \|g\|_Q \leq 1 \right\}
= \sup \left\{ \left| \int_G \varphi(s)\tilde{f} * g(s) \, ds \right| : g \in L^1(G), \|g\|_Q \leq 1 \right\}
\leq \|f\|_1 \sup \left\{ \left| \int_G \varphi(s)g(s) \, ds \right| : g \in L^1(G), \|g\|_Q \leq 1 \right\} = \|f\|_1 \|\varphi\|_{M_2}
$$

where $\tilde{f}(t) = \Delta(t^{-1})f(t^{-1})$. Hence $f * \varphi \in M_2(G)$. \hfill \Box

The fact that Hilbert spaces, which are exactly the $[\text{SQ}_2]$-spaces, are also $[\text{SQ}_p]$ spaces (see [17]) gives a contractive embedding $M_2(G) \hookrightarrow M_p(G)$.

The following construction is modelled after Proposition 1.3 of [9].

**Lemma 3.2.** Fix $a$ in $A_{p,c}(G)$ with $a \geq 0$ and $\int_G a = 1$. For $T$ in $CV_p(G)$ and $\omega$ in $N^p(G)$ let

$$q_{T,\omega} \in M_2(G)^* \text{ be given by } q_{T,\omega}(\varphi) = \langle T, (a * \varphi) \cdot \omega \rangle.$$

Then $q_{T,\omega} \in Q(G)$.

**Proof.** It is evident that $\|q_{T,\omega}\|_{M_2} \leq \|T\|\|\omega\|_{N^p(G)}$. Hence by norm approximation, it suffices to show that if $\omega \in N_2^p(G)$, say $\omega \in N^p(K)$ for some $K$ in $K(G)$, then $q_{T,\omega}(\varphi) = \int_K \varphi g$ for some $g$ in $L^1(G)$. We may further assume $K$ is so large that $a \in A_p(K)$. With these assumptions, let
$S = \text{supp}(a)^{-1}\text{supp}(P\omega)$, and we have

$$\begin{align*}
(a \ast \varphi)(s)P\omega(s) &= \int_G a(t)\varphi(t^{-1}s)P\omega(s) \, dt \\
&= \int_G a(t)1_S(t^{-1}s)\varphi(t^{-1}s)P\omega(s) \, dt \\
&= \int_G 1_S(t^{-1})\varphi(t^{-1})a_t(s)P\omega(s) \, dt.
\end{align*}$$

Let $L = KK^{-1}S^{-1} \cup KK^{-1}$ in $\mathcal{K}(G)$, which contains the support of each $a_tP\omega$. We note that $t \mapsto a_t : S^{-1} \to A_p(L)$ is continuous. Indeed, on elementary tensors $\xi \otimes f$ in $N^p(K)$, we have that $(\xi \ast \hat{f})_t = \xi \ast (\lambda_p(t)f)'$ and left translation is continuous on elements of $L^p(G)$. Hence we realize

$$(a \ast \varphi)P\omega = \int_G 1_S(t^{-1})\varphi(t^{-1})[a_t P\omega] \, dt$$

as a Bochner integral in $A_p(L)$, i.e. respecting the norm $\| \cdot \|_{A_p(L)}$. Then let

$$g(t) = 1_S(t)\Delta(t^{-1})\langle T, a_{t^{-1}} P\omega \rangle$$

which defines an element of $L^1(G)$. We may thus use the dual pairing (2.2) to compute

$$\int_G \varphi g = \int_G 1_S(t^{-1})\varphi(t^{-1})\langle T, a_t P\omega \rangle \, dt = \langle T, \int_G 1_S(t^{-1})\varphi(t^{-1})[a_t P\omega] \rangle = \langle T, (a \ast \varphi) \cdot \omega \rangle.$$

This establishes the desired claim.

**Proof of Theorem 1.1.** Let $a$ be as in Lemma 3.2 and $(\varphi_i) \subset A_{2,c}(G)$ be a net converging weak* to 1. Note that $A_{2,c}(G) \subseteq \mathcal{M}_2(G) \subseteq \mathcal{M}_p(G)$, and hence $A_{2,c}(G)$ is a space of compactly supported elements of $\mathcal{M}_p(G)$. The argument just before Corollary 2.4 tells us that any compactly supported element of $\mathcal{M}_p(G)$ is in $A_{p,c}(G)$. Since $A_p(G)$ is a homogeneous space for left translations, we find that each $a \ast \varphi_i \in A_p(G)$.

Given $T$ in $CV_p(G)$, let $q_{T,\omega}$ be as in Lemma 3.2. Then we see that

$$\langle (a \ast \varphi_i) \cdot T, \omega \rangle = \langle T, (a \ast \varphi_i) \cdot \omega \rangle = q_{T,\omega}(\varphi_i)$$

$$\xrightarrow{\text{i}} q_{T,\omega}(1) = \langle T, (a \ast 1) \cdot \omega \rangle = \langle T, \omega \rangle$$

since $a \ast 1 = \left[ \int_G a \right] 1 = 1$, by assumption on $a$. But Corollary 2.8 provides that each $(a \ast \varphi_i) \cdot T \in PM_p(T)$, and hence so too the weak* limit of this net, $T$, is also in $PM_p(G)$. 


Hints of the Theorem 1.1 are given by Cowling [3]. Given how heavily
we have exploited his results, i.e. Theorem 2.2, we suspect he knew how to
prove this.

**Remark 3.3.** The space $\mathcal{M}_p(G)$ generally has a predual $Q_p(G)$, as was
verified by T. Miao in an unpublished manuscript. Using this, An, Lee and
Ruan [11] defined the $p$-approximation property ($p$-AP), which is having 1 in
the weak* closure of $A_p(G)$ in $\mathcal{M}_p(G)$. As verified by Vergara [23], $p$-AP is
equivalent to $p'$-AP, and if $2 \leq p \leq q < \infty$, then $p$-AP implies $q$-AP. Hence
$p$-AP for $p \neq 2$ is ostensibly more general than approximation property (2-
AP). However, it is shown in [23] that certain canonical higher rank simple
Lie groups fail $p$-AP for any $p$ – the same ones known to Lafforgue and de la
Salle [18] and Haagerup and de Laat [11] – giving convincing evidence that,
at least for connected groups and lattices within, that $p$-AP is equivalent
to 2-AP. No examples are known of groups admitting $p$-AP for some $p$, but
not 2-AP.

Nonetheless, it is simple to modify the Lemmas of this section to acco-
modate the assumption of $p$-AP. We refer the reader [23] for details.

### 4 Proof of the commutation results, Theorem 1.2

Theorem 2.2 above, provides a very useful approximation of a convoluter $T$
in $CV_p(G)$ by a bounded net of operators $(\lambda_p(h_i))$ where each $h_i \in CV_p(G)$. If we were conducting approximations for $p = 2$, then we need only consider self adjoint $T = T^*$ and we need not worry about determining the structure of $\lambda_p(h_i)^*$. For $p \neq 2$ we have no such luck. We overcome this difficulty below. Furthermore, in the spirit of Theorem 2.2 we give a careful pointwise approximation of a convoluter, on a dense subspace, by convolutions with elements of $L^1(G)$.

We note that $f \mapsto \tilde{f}$, where $\tilde{f}(t) = \Delta(t^{-1})f(t^{-1})$ for (locally) a.e. $s$, defines an isometric involution on $L^1(G)$, and clearly defines a linear map on $L^1_{loc}(G)$ which satisfies $(h * f)^* = \tilde{f} * \tilde{h}$ for any convolvable pair of elements. Hence if $(f_i)$ is a contractive approximate identity on $L^1(G)$, then so too is $(\tilde{f_i})$. Furthermore, if $f \in L^1(G)$, then $\lambda_p(f)^* = \lambda_{p'}(\tilde{f})$, as is standard to check. We also observe that since $\rho_p(s)^* = \rho_{p'}(s^{-1})$ for all $s$ in $G$, we have \{$T^* : T \in CV_p(G)$\} = $CV_{p'}(G)$.

**Theorem 4.1.** Let $T \in CV_p(G)$.
(i) Then there is a net of elements \((k_i)\) in \(CV_p^p(G)\) for which \((\lambda_p(k_i))\) is a bounded net in \(CV_p(G)\) converging strong operator to \(T\), and also \((\tilde{k}_i)\) is a net in \(CV_p^{p'}(G)\) for which \((\lambda_{p'}(\tilde{k}_i))\) is bounded in \(CV_{p'}(G)\) and converges to \(T^*\).

(ii) Given \(g, f\) in \(C_c(G)\) and \(\varepsilon > 0\), there is \(k\) in \(C_c(G)\) for which

\[
\|Tf - \lambda_p(k)f\|_p < \varepsilon \text{ and } \|T^*g - \lambda_{p'}(\tilde{k})g\|_{p'} < \varepsilon.
\]

Here \(k\) is dependant upon the choices of \(g, f\) and \(\varepsilon\).

Proof. (i) As in Theorem 2.2 we let \((f_i) \subset C_c(G)\) be a contractive approximate identity for \(L^1(G)\) and set \(h_i = T f_i\). We note that

\[
\lambda_p(h_i) = \lambda_p(T f_i) = T \lambda_p(f_i).
\]

Let \(g, f \in C_c(G)\). Since \(h_i \in L^1_{loc}(G)\), \(h_i * g \in L^1_{loc}(G)\) and, letting \(S = \text{supp} f \cup \text{supp} g\), we have

\[
\int_G \tilde{h}_i * g(s) f(s) ds = \int_S \int_{SS^{-1}} \Delta(t^{-1}) h_i(t^{-1}) g(t^{-1}s) f(s) dt ds
\]

\[
= \int_{SS^{-1}} \int_S g(s) h_i(t) f(t^{-1}s) ds dt
\]

\[
= \langle g, \lambda_p(h_i)f \rangle = \langle g, T \lambda_p(f_i)f \rangle = \langle \lambda_{p'}(\tilde{f}_i) T^*g, f \rangle.
\]

Hence by density it follows that \(\tilde{h}_i * g = \lambda_{p'}(\tilde{f}_i) T^*g\), and hence \(\tilde{h}_i \in CV_p^1(G)\) with

\[
\lambda_p(\tilde{h}_i) = \lambda_{p'}(\tilde{f}_i) T^*.
\]

We do not know of a means to see that \(\tilde{h}_i \in CV_p^{p'}(G)\). We use another convolution to overcome this difficulty.

Let \(k_i = f_i * h_i\). It is clear that \(k_i \in CV_p^p(G)\) with \(\lambda_p(k_i) = \lambda_p(f_i) \lambda_p(h_i)\), and hence \(\|\lambda_{p'}(\tilde{k}_i)\| \leq \|f_i\|_1 \|\lambda_p(h_i)\|\). Likewise \(\tilde{k}_i = \tilde{h}_i * \tilde{f}_i \in CV_p^1(G)\) with \(\lambda_{p'}(\tilde{k}_i) = \lambda_{p'}(\tilde{h}_i) \lambda_p(\tilde{f}_i)\), with \(\|\lambda_{p'}(\tilde{k}_i)\| \leq \|\lambda_{p'}(\tilde{h}_i)\| \|f_1\|_1\). Moreover, as \(T^* \in CV_{p'}(G)\) we have that \(T^*(\tilde{f}_i) \in CV_p^{p'}(G)\) and

\[
\lambda_{p'}(\tilde{k}_i) = \lambda_{p'}(\tilde{h}_i) \lambda_p(\tilde{f}_i) = \lambda_{p'}(\tilde{f}_i) T^* \lambda_p(\tilde{f}_i) = \lambda_{p'}(\tilde{f}_i * T^* \tilde{f}_i)
\]

so \(\tilde{k}_i = \tilde{f}_i * T^*(\tilde{f}_i) \in CV_p^{p'}(G)\) too. We have that

\[
\|\lambda_p(k_i)f - Tf\|_p = \|f_i * [T f_i * f] - Tf\|_p
\]

\[
\leq \|f_i * [T f_i * f] - f_i * Tf\|_p + \|f_i * Tf - Tf\|_p
\]

\[
\leq \|f_i\|_1 \|T f_i * f - Tf\|_p + \|f_i * Tf - Tf\|_p
\]
which tends to zero. By uniform boundedness of the operators \( \lambda_p(k_i) \) and density of \( C_c(G) \) in \( L^p(G) \), we see that \( \lambda_p(k_i) \) tends to \( T \) in the strong operator topology. Similarly, the net \( (\lambda_{p'}(\delta_k)) \) is bounded and tends to \( T^* \) in the strong operator topology.

(ii) Let \( \delta > 0 \). Let \( i \) be such that

\[
\| T f - k_i * f \|_p < \delta \quad \text{and} \quad \| T^* g - \delta_k * g \|_{p'} < \delta.
\]

Inner regularity of the Haar measure provides \( K \) in \( K(G) \) for which

\[
\| k_i - 1_K k_i \|_p < \delta \quad \text{and} \quad \| \delta_k - 1_K \delta_k \|_{p'} < \delta.
\]

Let \( k = 1_K k_i \) which is in \( L^1 \cap L^p(G) \), and note that \( \delta_k = 1_K \delta_k \). Since \( l * f = \rho_p(\Delta^{-1/p} \delta_k)l \) for \( l \) in \( L^p(G) \), we have that

\[
\| T f - k * f \|_p \leq \| T f - k_i * f \|_p + \| k_i * f - k * f \|_p < \delta + \| \rho_p(\Delta^{-1/p} \delta_k) \|_{p'} \delta.
\]

Likewise we have that

\[
\| T^* g - \delta_k * g \|_{p'} < \delta + \| \rho_p(\Delta^{-1/p} \delta_k) \|_{p'} \delta.
\]

Hence an obvious choice of \( \delta \) yields the desired inequalities. \( \square \)

We note that the sorts of global estimates found in (i), above, were required to obtain the pointwise estimates of (ii).

**Remark 4.2.** As noted in the introduction, there is a self-inverting isometry \( U \) on \( L^p(G) \), which intertwines \( \lambda_p \) and \( \rho_p \): \( U \lambda_p(s) = \rho_p(s) U \) for \( s \) in \( G \). Hence \( U^* \) intertwines \( \lambda_{p'} \) and \( \rho_{p'} \). We let \( CV_p(G) = \lambda_p(G)' \) so

\[
CV_p(G) = (U \rho_p(G) U)' = U \rho_p(G)' U = U CV_{p'}(G) U.
\]

If \( S \in CV_{p'}(G) \) and \( g, f \in C_c(G) \), then \( U^* g, U f \in C_c(G) \). Hence, given \( \varepsilon > 0 \) Theorem 4.1 (ii) provides \( k' \) in \( L^1(G) \) for which

\[
\| USUU f - \lambda_p(k') U f \|_p < \varepsilon \quad \text{and} \quad \| (USU)^* U^* g - \lambda_{p'}(\delta_k) U^* g \|_{p'} < \varepsilon
\]

hence multiplying the arguments of the expressions by the respective isometries \( U \) and \( U^* \) we obtain

\[
\| S f - \rho_p(k') f \|_p < \varepsilon \quad \text{and} \quad \| S^* g - \rho_{p'}(\delta_k) g \|_{p'} < \varepsilon.
\]

The key reason we have Theorem 4.1 is that for \( k, k' \) in \( L^1(G) \) we have \( \lambda_p(k) \rho_p(k') = \rho_p(k') \lambda_p(k) \). We do not know how to verify directly that \( \lambda_p(h) \rho_p(h') = \rho_p(h') \lambda_p(h) \) for \( h, h' \) in \( CV_p(G) \).
Proof of Theorem 1.2. Let $PM_p'(G) = U PM_p(G)U = \overline{\text{int}^{w^*}} \rho_p(G)$. We have that

$$PM_p(G)' = \lambda_p(G)' = CV_p'(G) \supseteq PM_p'(G)$$

and $PM_p'(G)' = CV_p(G)$. Thus it follows that

$$PM_p(G)'' = CV_p'(G)' \subseteq PM_p'(G)' = CV_p(G).$$

Hence we will be done once we show that $CV_p(G) \subseteq CV_p'(G)'$, i.e. if $T \in CV_p(G)$ and $S \in CV_p'(G)$, we wish to see that $TS = ST$.

To this end, fix $g, f$ in $C_c(G)$. As in Theorem 4.1 (ii) and Remark 4.2, given $\varepsilon > 0$, find $k, k'$ in $L^1(G)$ for which

$$\|Tf - \lambda_p(k)f\|_p < \varepsilon \quad \text{and} \quad \|T^* g - \lambda_p(k')g\|_{p'} < \varepsilon$$

and

$$\|Sf - \rho_p(k')f\|_p < \varepsilon \quad \text{and} \quad \|S^* g - \rho_p(k')g\|_{p'} < \varepsilon.$$

Then

$$|\langle g, TSf \rangle - \langle g, \lambda_p(k)\rho_p(k')f \rangle|$$

$$\leq |\langle T^*g, Sf - \rho_p(k')f \rangle| + |\langle T^*g - \lambda_p(k')g, \rho_p(k')f \rangle|$$

$$\leq \|T\|\|g\|_{p'}\varepsilon + \varepsilon\|\rho_p(k')f\|_p \leq (\|T\|\|g\|_{p'} + \|S\|\|f\|_p + \varepsilon)\varepsilon$$

and, by a similar calculation, we see that

$$|\langle g, STf \rangle - \langle g, \rho_p(k')\lambda_p(k)f \rangle| \leq (\|S\|\|g\|_{p'} + \|T\|\|g\|_p + \varepsilon)\varepsilon.$$

Since $\lambda_p(k)\rho_p(k') = \rho_p(k')\lambda_p(k)$ for any $k, k'$ in $L^1(G)$, we can take $\varepsilon$ to 0 and see that

$$\langle g, TSf \rangle = \langle g, STf \rangle.$$

Hence by density we see that $\langle \xi, TSf \rangle = \langle \xi, STf \rangle$ for any $\xi$ in $L^p(G)$ and $f$ in $L^p(G)$, so $TS = ST$, as desired.

We obtain an “elementary” proof of Dixmier’s result [8], i.e. without use of of left Hilbert algebras. The common algebra in the result below is usually denoted $VN(G)$ and called the group von Neuman algebra.

**Corollary 4.3.** We have that

$$CV_2(G) = CV_2'(G)' = PM_2(G) = PM_2'(G)'.$$

**Proof.** Here $\lambda_2(G)$ is a self-adjoint subset of $\mathcal{B}(L^2(G))$, so $VN(G) = PM_2(G)$ is self-adjoint. The bicommutant theorem of von Neumann, [22 II.3.9] for example, tells us that $PM_2(G)''$ is the $\sigma$-strong* closure of $PM_2(G)$. As a convex subspace of $\mathcal{B}(L^2(G))$ is weak*-closed if and only if it is $\sigma$-strong* closed (see [22 II.2.6]), we see that $PM_2(G)'' = PM_2(G)$. The rest follows from the theorem above.


Acknowledgements

The second named researcher was partially supported by an NSERC Discovery Grant.

References


