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A $p$-adic analogue of Siegel’s theorem on sums of squares

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Abstract
Siegel proved that every totally positive element of a number field $K$ is the sum of four squares, so in particular the Pythagoras number is uniformly bounded across number fields. The $p$-adic Kochen operator provides a $p$-adic analogue of squaring, and a certain localisation of the ring generated by this operator consists of precisely the totally $p$-integral elements of $K$. We use this to formulate and prove a $p$-adic analogue of Siegel’s theorem, by introducing the $p$-Pythagoras number of a general field, and showing that this number is uniformly bounded across number fields. We also generally study fields with finite $p$-Pythagoras number and show that the growth of the $p$-Pythagoras number in finite extensions is bounded.

KEYWORDS
Kochen operator, number fields, $p$-valuations

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11E25, 11S99, 11U09, 12D15

1 | INTRODUCTION

The study of sums of squares has a long history. In the context of the integers, Fermat, Euler, Lagrange and many others studied which integers are a sum of a certain number of square integers. The possibly most famous result in this direction is Lagrange’s Four Squares Theorem [13, Thm. 369] that every nonnegative integer is the sum of four squares. In fact, earlier Euler had proved a version of this theorem for $\mathbb{Q}$: every nonnegative rational number is the sum of four square rational numbers. A comprehensive history of these theorems may be found in [6, Chapter VIII]. In the other direction, for both $\mathbb{Z}$ and $\mathbb{Q}$ there exist nonnegative numbers that cannot be written as a sum of three squares. The Pythagoras number $\pi(F)$ of a field $F$ is the smallest $n$ such that

$$\{ x^2_1 + \cdots + x^2_m \mid x_1, \ldots, x_m \in F, m \in \mathbb{N} \} = \{ x^2_1 + \cdots + x^2_n \mid x_1, \ldots, x_n \in F \}.$$

Using this terminology, Euler’s theorem becomes the statement that $\pi(\mathbb{Q}) = 4$. The following generalization of Euler’s theorem was conjectured by Hilbert and proven by Siegel in [25], cf. [20, Ch. 7, §1, 1.4]:

**Theorem 1.1 (Siegel).** For all number fields $F$, $\pi(F) \leq 4$.

The study of the Pythagoras number of a field is intimately related to the study of the orderings on that field, since by a theorem of Artin and Schreier the sums of squares are precisely the totally positive elements. In a number field $F$, these can be described simply as those elements that are mapped to $\mathbb{R}_{\geq 0}$ by every embedding of $F$ into $\mathbb{R}$, cf. [20, Ch. 3 and 7].
We define and study a \( p \)-adic version of the Pythagoras number, namely the \( p \)-Pythagoras number \( \pi_p(F) \) of a field \( F \), or more generally the \((p, \tau)\)-Pythagoras number, see Section 2.2 for the definition. Just like the Pythagoras number gives information on the set of totally positive elements, the \( p \)-Pythagoras number relates to the set of totally \( p \)-integral elements, which in a number field \( F \) can be described simply as those elements that are mapped to \( \mathbb{Z}_p \) by every embedding of \( F \) into \( \mathbb{Q}_p \). Our main result is an inexplicit analogue of Siegel’s theorem:

**Theorem 1.2.** Let \( p \) be a prime number. There exists \( N_p \in \mathbb{N} \) such that \( \pi_p(F) \leq N_p \) for every number field \( F \).

This result will be deduced from the more general Theorem 4.9. We also give some general results on fields \( F \) with finite \((p, \tau)\)-Pythagoras number and prove in Theorem 5.9 that the growth of the \((p, \tau)\)-Pythagoras number is bounded in finite extensions. As an application, we show in Corollary 6.5 that for every open-closed subset of the \( p \)-adic spectrum of \( F \), the associated holomorphy ring is diophantine. A further application can be found in the forthcoming work [2], in which we use the results of this paper to show that rings of formal power series over number fields are \( \mathbb{Z} \)-diophantine in their quotient fields.

## 2 | THE \( (p, \tau) \)-PYTHAGORAS NUMBER

### 2.1 | \( p \)-valuations

A (Krull) valuation \( v \) on a field \( F \) is a \( p \)-valuation if it has a finite residue field \( \bar{F}_v \) of characteristic \( p \) and value group \( v(F^\times) \) such that the interval \((0, v(p)] \) is finite. A (finite) prime \( \mathfrak{P} \) of a field \( F \) is an equivalence class of \( p \)-valuations on \( F \) (for the usual notion of equivalence of valuations), for some prime number \( p \). We write \( v_{\mathfrak{P}} \) for a representative of \( \mathfrak{P} \) which has \( \mathbb{Z} \) as smallest non-trivial convex subgroup of the value group. See [22] for basics regarding \( p \)-valuations, and [10] for details on this notion of prime and some of the following definitions.

**Example 2.1.** The primes of a number field \( K \) correspond precisely to the finite places in the usual sense and we will identify them. If \( K = \mathbb{Q} \) and \( p \) is a prime number then \( v_p \) denotes the usual \( p \)-adic valuation, and we denote the corresponding prime also by \( p \).

For the rest of this work we fix a triple \((K, p, \tau)\), where \( K \) is a number field, \( p \) is a finite prime of \( K \), and \( \tau \) is a pair of natural numbers \((e, f) \in \mathbb{N}^2 \). We denote by \( t_p \) a uniformizer of \( v_p \), i.e. an element with \( v_p(t_p) = 1 \), we let \( q \) denote the size of the residue field \( \bar{K}_{v_p} \).

For a field extension \( F/K \) with \( \mathfrak{P} \) a prime of \( F \) lying above \( p \), the relative initial ramification is \( e(\mathfrak{P}|p) := v_p(t_p) \), the relative residue degree is \( f(\mathfrak{P}|p) := \left[ \bar{F}_{v_p} : \bar{K}_{v_p} \right] \), and the pair \((e(\mathfrak{P}|p), f(\mathfrak{P}|p)) \) is the relative type of \( \mathfrak{P} \) over \( p \). We say \( \mathfrak{P} \) is of relative type at most \( \tau \) if \( e(\mathfrak{P}|p) \) is no greater than \( e \), and \( f(\mathfrak{P}|p) \) divides \( f \). Likewise, for \( \tau' = (e', f') \) we write \( \tau \leq \tau' \) if \( e \leq e' \) and \( f \mid f' \). We denote by \( S(F) \) the set of primes of \( F \), by \( S_p(F) \subseteq S(F) \) the set of those primes \( \mathfrak{P} \) of \( F \) lying above \( p \), and by \( S_{p e}^e(F) \subseteq S_p(F) \) the subset of those primes \( \mathfrak{P} \) of \( F \) which are of relative type at most \( \tau \) over \( p \). The corresponding holomorphy ring is

\[
R_{p e}^e(F) := \bigcap_{\mathfrak{P} \in S_{p e}^e(F)} \mathcal{O}_{\mathfrak{P}},
\]

where \( \mathcal{O}_{\mathfrak{P}} \) is the valuation ring of \( \mathfrak{P} \), and

\[
\Gamma_{p e}^e(F) := \left\{ \frac{a}{1 + t_p b} \mid a, b \in \mathcal{O}_p, \left(\frac{\gamma_{p e}^e}{\mathcal{O}_p}(F), 1 + t_p b \neq 0 \right) \right\}
\]

is the corresponding Kochen ring, where

\[
\gamma_{p e}^e(X) := \frac{1}{t_p} \left( \frac{X^{q_f} - X}{(X^{q_f} - X)^2 - 1} \right)^e
\]

is the Kochen operator. Here and in what follows, if \( \gamma \in F(X) \) is a rational function, we mean by \( \gamma(F) \) the image of \( \gamma \) on \( F \setminus \{ \text{poles of } \gamma \} \). Note that \( \Gamma_{p e}^e(F) \) does not depend on the choice of \( t_p \), since the quotient of two uniformizers of \( v_p \) is an element of \( \mathcal{O}_p^e \). Recall that \( R_{p e}^e(F) \) is the integral closure of \( \Gamma_{p e}^e(F) \), with equality in the case \( e = 1 \), see [22, Cor. 6.9] and the subsequent discussion for more details.
Example 2.2. If \( p \) is any place of the number field \( K \), we denote by \( K_p \) the completion of \( K \) with respect to \( p \). If \( p \) is a finite place, then \( K_p \) is a non-archimedean local field and \( p \) extends to a unique prime \( \mathfrak{p} \) of \( K_p \) of the same type, so \( R_p(K_p) = R_p^{[1,1]}(K_p) = \mathcal{O}_p \). In fact, any non-archimedean local field \( E \) of characteristic zero carries a unique prime, whose valuation ring we denote by \( \mathcal{O}_E \), cf. [22, Thm. 6.15]. We say that an extension of non-archimedean local fields is of relative type at most \( \tau \) if this is true for the respective primes.

The real holomorphy ring of \( F \) is the intersection of the positive cones of the orderings on \( F \), i.e. the set of elements that are nonnegative under every ordering on \( F \). By the theorem of Artin and Schreier it can alternatively be described as the set of sums of squares, and the classical Pythagoras number may be seen as a measure of the complexity of this description in terms of squares. The holomorphy ring \( R_\tau(F) \) is defined above as an intersection of the valuation rings of certain \( p \)-valuations, and it also equals the integral closure of \( \Gamma_\tau(F) \). Thus a \( p \)-adic analogue of the Pythagoras number should somehow measure the complexity of the description of \( R_\tau(F) \) in terms of the rational function \( \gamma_{p,\tau}^\tau \). We now define such a \( p \)-adic analogue.

2.2 | The (\( \mathfrak{p}, \tau \))-Pythagoras number

Let \( F/K \) be an extension. For \( g \in \mathcal{O}_p[X_1, \ldots, X_n] \), we write

\[
R^\tau_{p,\mathfrak{p},\tau}(F) := \left\{ \frac{a}{1 + t_p b} \mid a, b \in g\left(\gamma_{p,\mathfrak{p}}^\tau(F), \ldots, \gamma_{p,\mathfrak{p}}^\tau(F)\right), 1 + t_p b \neq 0 \right\},
\]

and for \( n \geq 1 \)

\[
R^\tau_{p,\mathfrak{p},\tau,n}(F) := \left\{ x \in F \mid x^m + a_m x^{m-1} + \cdots + a_0 = 0 \text{ with } 1 \leq m \leq n, a_0, \ldots, a_m \in R^\tau_{p,\mathfrak{p},\tau}(F) \right\}.
\]

We denote by \( \mathcal{P}_p \) the finite set of those \( g \in \mathcal{O}_p[X_1, \ldots, X_n] \) of degree and height at most \( n \) (cf. [4, Def. 1.6.1]). We write

\[
R^\tau_{p,\mathfrak{p}}(F) := \bigcup_{g \in \mathcal{P}_p} R^\tau_{p,\mathfrak{p},\tau,n}(F),
\]

where \( t_p \) varies over those (finitely many) elements of the ring of integers \( \mathcal{O}_K \) which are uniformizers for \( \mathfrak{p} \) of minimal height. Then \( (R^\tau_{p,n}(F))_{n \in \mathbb{N}} \) is an increasing chain of subsets of \( F \) and

\[
R^\tau_{p}(F) = \bigcup_{n \in \mathbb{N}} R^\tau_{p,n}(F).
\]

The (\( \mathfrak{p}, \tau \))-Pythagoras number \( \pi^\tau_{\mathfrak{p}}(F) \) of \( F \) is the smallest \( n \) such that

\[
R^\tau_{p}(F) = R^\tau_{p,n}(F),
\]

and we write \( \pi^\tau_{\mathfrak{p}}(F) = \infty \) if there is no such \( n \). In other words,

\[
\pi^\tau_{\mathfrak{p}}(F) := \inf \left\{ n \in \mathbb{N} \mid R^\tau_{p,n}(F) = R^\tau_{p,n}(F) \right\} \in \mathbb{N} \cup \{ \infty \}.
\]

In the case \( K = \mathbb{Q} \), \( \mathfrak{p} = p \) and \( \tau = (1, 1) \), we write \( R_p(F) \) and \( \pi_p(F) \), omitting the relative type \((1,1)\), and we speak of the \( p \)-Pythagoras number. We also write \( \gamma_p := \gamma_p^{(1,1)} \), and note that the only two uniformizers (of the prime \( p \)) in \( \mathbb{Z} \) of minimal height are \( p \) and \( -p \), with \( \gamma_p^{(1,1)} = -\gamma_p \). We discuss some possible variations of our definition of the (\( \mathfrak{p}, \tau \))-Pythagoras number in Remarks 3.11 and 3.12.

Example 2.3. Since \( \mathbb{C} \) is algebraically closed and carries no \( p \)-valuation, we have

\[
R_p(\mathbb{C}) = \mathbb{C} = \gamma_p(\mathbb{C}),
\]

in particular \( \pi_p(\mathbb{C}) = 1 \).

Example 2.4. It follows easily from Hensel’s lemma that

\[
R_p(\mathbb{Q}_p) = \mathbb{Z}_p = \gamma_p(\mathbb{Q}_p),
\]

in particular \( \pi_p(\mathbb{Q}_p) = 1 \), see [22, Thm. 6.15].
Example 2.5. In [11, Lem. 3.02] it is shown that every so-called pseudo $p$-adically closed field $F$ (where pseudo $p$-adically closed means that a certain geometric local-global principle holds for varieties over $F$) satisfies

$$R_p(F) = \gamma_p(F) + \gamma_p(F) + \gamma_p(F),$$

hence $\pi_p(F) \leq 3$. This applies for example to the field $\mathbb{Q}^{1/p}$ of totally $p$-adic algebraic numbers by a result of Moret–Bailly [17], where the local-global principle takes the following simple form: If $V$ is a geometrically irreducible smooth variety over $\mathbb{Q}^{1/p}$ which has a $Q_p$-rational point for every embedding of $\mathbb{Q}^{1/p}$ into $Q_p$, then it has a $\mathbb{Q}^{1/p}$-rational point.

It is known that there are fields $F$ with $\pi(F) = \infty$, for example $F = \mathbb{R}(x_1, x_2, \ldots)$, see [15, Ch. XI, Example 5.9(5)]. On the other hand, we do not know if $\pi_p(F) = \infty$ for any field:

Question 2.6. Is $\pi_p(Q(X_1, X_2, \ldots)) = \infty$?

2.3 Explicit bounds and uniformity in $p$

We now prove a few rather elementary statements about $\pi_p(Q)$. We will drop the relative type $\tau = (1, 1)$ from all notation. Let $\ell'$ be a prime number distinct from $p$.

**Lemma 2.7.** We have $\gamma_p(Q) \subseteq Z(\ell')$ if and only if neither $X^p - X + 1$ nor $X^p - X - 1$ has a zero in $F_\ell$.

**Proof.** Let $x \in Q$, recall that $\gamma_p(x) = \frac{1}{p}((x^p - x) - (x^p - x)^{-1})^{-1}$ and denote by $v_\ell$ the $\ell$-adic valuation. If $v_\ell(x^p - x) < 0$ or $v_\ell(x^p - x) > 0$, then $v_\ell(\gamma_p(x)) > 0$. If $v_\ell(x^p - x) = 0$, then $x \in Z(\ell)$, and $v_\ell(\gamma_p(x)) < 0$ if and only if $(x^p - x) - (x^p - x)^{-1} \equiv 0 \mod \ell'$, which means that $x^p - x \equiv \pm 1 \mod \ell'$. □

**Proposition 2.8.** $Z[\gamma_p(Q)] \subseteq Z(\ell')$.

**Proof.** There exists a prime number $\ell' \neq p$ such that $Z[\gamma_p(Q)]$ is contained in $Z(\ell')$ by Lemma 2.7: specifically, the criterion given there is satisfied by $\ell' = 2$ if $p$ is odd and by $\ell' = 17$ for $p = 2$. □

**Lemma 2.9.** If $\ell' - 1 | p - 1$ then $\gamma_p(Q) \subseteq \ell'Z(\ell')$.

**Proof.** If $\ell' - 1 | p - 1$, then $x^p - x = 0$ for all $x \in F_\ell$. Thus $v_\ell(\gamma_p(x)) > 0$ for all $x \in Q$, where $v_\ell$ is the $\ell$-adic valuation. □

**Proposition 2.10.** For every finite set $P \subseteq Q(X_1, X_2, \ldots)$, there exist some $p$ and $\ell' \neq p$ with

$$\bigcup_{g \in P} R_{p,g,p}(Q) \subseteq Z(\ell').$$

In particular, $\sup_p \pi_p(Q) = \infty$.

**Proof.** Choose $\ell' > |P| + 1$ such that $P \subseteq Z(\ell')[X_1, X_2, \ldots]$. There exists $a \in Z$ such that $a \neq 0 \mod \ell'$ and $a \equiv g(0, \ldots, 0) \mod \ell'$ for every $g \in P$. By Dirichlet’s theorem on primes in arithmetic progressions (see [18, VII, (13.2)]), there exist infinitely many primes $p > \ell'$ with $p \equiv 1 \mod \ell' - 1$ and $p \equiv -a^{-1} \mod \ell'$. Then

$$g(\gamma_p(Q), \ldots, \gamma_p(Q)) \subseteq g(0, \ldots, 0) + \ell'Z(\ell')$$

by Lemma 2.9, hence $1 + pg(\gamma_p(Q), \ldots, \gamma_p(Q)) \subseteq Z(\ell')$ by the choice of $a$ and $p$. Thus $R_{p,g,p}(Q) \subseteq Z(\ell')$ for every $g \in P$.

By the integral closedness of $Z(\ell')$ this implies $R_{p,g,p,n}(Q) \subseteq Z(\ell')$ for every $n$. Note that $R_{p,g,-p,n}(F) = -R_{p,g,-p,n}(F)$, where $g^*(X_1, \ldots, X_n) = -g(-X_1, \ldots, -X_n)$ has the same height as $g$. Therefore, applying the above to the set $P$ of all $f \in Q[X_1, \ldots, X_n]$ of degree and height at most $n$, we obtain $\ell'$ and $p > \ell'$ with

$$\bigcup_{g \in P} (R_{p,g,p,n}(F) \cup R_{p,g,-p,n}(F)) \subseteq \bigcup_{p \in P} R_{p,g,p,n}(F) \subseteq Z(\ell'),$$

and therefore $\pi_p(Q) > n$. □

2.4 The Kochen operator

For later use, we explore several simple properties of the Kochen operator. Let $F/K$ be any extension.
Lemma 2.11. Let \( \mathfrak{B} \in S^\tau_p(F) \) and suppose that \( x \in F \) is not a pole of \( \gamma^\tau_{p,\mathfrak{B}} \). Then

\[
\nu_q(x^\tau_{p,\mathfrak{B}}(x)) = \begin{cases} 
-e v_q(x) - \nu_q(t_p) & \text{if } \nu_q(x) < 0, \\
ev_q(x) - \nu_q(t_p) & \text{if } \nu_q(x) > 0, \\
\nu_q(x^q f - x) - \nu_q(t_p) & \text{if } \nu_q(x) = 0 \text{ and } \nu_q(x^q - x) > 0,
\end{cases}
\]

\[
-e \nu_q\left((x^q f - x)^2 - 1\right) - \nu_q(t_p) & \text{if } \nu_q(x) = 0 \text{ and } \nu_q(x^q - x) = 0.
\]

Proof. This is a matter of calculating valuations.

Lemma 2.12. Let \( \mathfrak{B} \in S^\tau_p(F) \). Suppose that \( x \in F \) is not a pole of \( \gamma^\tau_{p,\mathfrak{B}} \) and satisfies either

(i) \( 0 < (e + 1)v_q(x) \leq v_q(t_p) \), or
(ii) \( v_q(x) = 0 \) and \( \left[ \mathbb{F}_q \left( \text{res}_q(x) \right) : \mathbb{F}_q \right] \nmid f \), where \( \text{res}_q(x) \) is the residue of \( x \).

Then

\[
\nu_q(x^\tau_{p,\mathfrak{B}}(x)) \leq -\frac{1}{e + 1} \nu_q(t_p) < 0.
\]

Proof. In case (i), Lemma 2.11 gives that

\[
\nu_q(x^\tau_{p,\mathfrak{B}}(x)) = e v_q(x) - \nu_q(t_p) \leq -\frac{1}{e + 1} \nu_q(t_p).
\]

In case (ii), the residue of \( x \) is not a root of \( X^q f - X \), and so

\[
\nu_q(x^\tau_{p,\mathfrak{B}}(x)) = -e \nu_q\left((x^q f - x)^2 - 1\right) - \nu_q(t_p) \leq -e \nu_q(t_p) \leq -\frac{1}{e + 1} \nu_q(t_p),
\]

also by Lemma 2.11.

Lemma 2.13. Let \( \mathfrak{B} \in S^\tau_p(F) \), let and \( x, y \in F \), and suppose that \( x \) is not a pole of \( \gamma^\tau_{p,\mathfrak{B}} \), and \( \nu_q(x^\tau_{p,\mathfrak{B}}(x)) < 0 \). If \( \nu_q(x - y) \geq \nu_q(t_p) \), then also \( y \) is not a pole of \( \gamma^\tau_{p,\mathfrak{B}} \), and \( \nu_q(x^\tau_{p,\mathfrak{B}}(y)) < 0 \).

Proof. If \( \nu_q(x) \leq 0 \), then in particular \( \nu_q(x) < v_q(t_p) \), while if \( v_q(x) > 0 \), then \( \nu_q(x^\tau_{p,\mathfrak{B}}(x)) = e v_q(x) - \nu_q(t_p) \) by Lemma 2.11, hence \( \nu_q(x^\tau_{p,\mathfrak{B}}(x)) < 0 \) implies that \( \nu_q(x) < v_q(t_p) \) also in this case. Therefore, in either case we conclude from \( \nu_q(x - y) \geq \nu_q(t_p) \) that \( \nu_q(x) = \nu_q(y) \). We make a case distinction:

Suppose first that \( \nu_q(x) \neq 0 \). By Lemma 2.11, in this case, \( \nu_q(x^\tau_{p,\mathfrak{B}}(x)) \) depends only on \( \nu_q(x) \). Therefore \( \nu_q(x^\tau_{p,\mathfrak{B}}(y)) = \nu_q(x^\tau_{p,\mathfrak{B}}(x)) < 0 \).

Suppose now that \( \nu_q(x) = 0 \). As \( x - y \) divides \( x^q f - y^q f \), we have that \( \nu_q(x^q f - y^q f + x) \geq \nu_q(x - y) \geq \nu_q(t_p) \). If \( \nu_q(x^q - x) = 0 \), then in particular \( \nu_q(x^q f - x) < \nu_q(t_p) \), while if \( \nu_q(x^q f - x) > 0 \), then \( \nu_q(x^\tau_{p,\mathfrak{B}}(x)) < 0 \) implies that \( \nu_q(x^q f - x) < \frac{1}{e} \nu_q(t_p) \leq \nu_q(t_p) \) by Lemma 2.11. Thus \( \nu_q(\gamma^\tau_{p,\mathfrak{B}}(y)) = \nu_q(x^q f - x) \) in both cases. If \( \nu_q(x^q f - x) = 0 \), then Lemma 2.11 gives immediately that \( \nu_q(x^\tau_{p,\mathfrak{B}}(y)) < 0 \), while if \( \nu_q(x^q f - x) > 0 \), then Lemma 2.11 shows that \( \nu_q(x^\tau_{p,\mathfrak{B}}(x)) \) depends only on \( \nu_q(x^q f - x) \), hence \( \nu_q(x^\tau_{p,\mathfrak{B}}(y)) = \nu_q(x^\tau_{p,\mathfrak{B}}(x)) < 0 \).

\[3 \ | \ DIOPHANTINE FAMILIES\]

A diophantine subset of a field \( F \) is the image of the \( F \)-rational points of some \( F \)-variety \( V \) under a morphism \( V \to \mathbb{A}^1_F \). As we want to discuss questions of uniformity we use the following slightly more sophisticated notion: An \( n \)-dimensional diophantine
family over $K$ is a map $D$ from the class of field extensions $F$ of $K$ to sets which is given by finitely many polynomials $f_1, \ldots, f_r \in K[X_1, \ldots, X_n, Y_1, \ldots, Y_m]$, for some $m$, in the sense that

$$D(F) = \{ x \in F^n \mid \exists y \in F^m : f_1(x, y) = 0, \ldots, f_r(x, y) = 0 \}$$

for every extension $F/K$. In this case, we say that the polynomials $f_1, \ldots, f_r$ define $D$. Note that if $E/F$ is an extension, then $D(F) \subseteq D(E)$.

**Remark 3.1.** From the point of view of algebraic geometry, an $n$-dimensional diophantine family $D$ over $K$ is given by a morphism of (not necessarily irreducible) $K$-varieties $\varphi : V \to \mathbb{A}_K^n$ in the sense that $D(F) = \varphi(V(F))$ for every extension $F/K$.

**Remark 3.2.** From the point of view of model theory, an $n$-dimensional diophantine family $D$ over $K$ is given by an existential formula $\varphi(x_1, \ldots, x_n)$ in the language of rings with free variables among $x_1, \ldots, x_n$ and parameters from $K$, in the sense that for every extension $F/K$, $D(F)$ is the set defined by $\varphi$ in $F$, i.e. the set of $a \in F^m$ such that $F \models \varphi(a)$. Such a formula is equivalent (modulo the theory of fields) to a formula of the form

$$\exists y_1 \ldots y_m : \bigwedge_{i=1}^r f_i(x_1, \ldots, x_n, y_1, \ldots, y_m) = 0$$

with $f_1, \ldots, f_r \in K[X_1, \ldots, X_n, Y_1, \ldots, Y_m]$.

Most of the usual constructions for diophantine sets (see e.g. [24]) go through for diophantine families:

**Lemma 3.3.** If $D_1$, $D_2$ are $n$-dimensional diophantine families over $K$, then there are $n$-dimensional diophantine families $D_1 \cup D_2$ and $D_1 \cap D_2$ over $K$ such that $(D_1 \cup D_2)(F) = D_1(F) \cup D_2(F)$ and $(D_1 \cap D_2)(F) = D_1(F) \cap D_2(F)$ for every $F/K$.

**Proof.** Suppose that the polynomials $f_1, \ldots, f_r \in K[X_1, \ldots, X_n, Y_1, \ldots, Y_m]$ define $D_1$ and that the polynomials $g_1, \ldots, g_s \in K[X_1, \ldots, X_n, Z_1, \ldots, Z_l]$ define $D_2$. We may assume that the variables $Y_j$ and $Z_j$ are distinct. We observe that $f_1, \ldots, f_r, g_1, \ldots, g_s$ define $D_1 \cup D_2$. Slightly less trivially, we have that $f_1g_1, \ldots, f_1g_j, \ldots, f_1g_s \in D_1 \cup D_2$.

**Lemma 3.4.** Suppose that $D_1$ and $D_2$ are $n_1$- respectively $n_2$-dimensional diophantine families over $K$. Then there is an $(n_1 + n_2)$-dimensional diophantine family $D_1 \times D_2$ over $K$ such that $(D_1 \times D_2)(F) = D_1(F) \times D_2(F)$ for every $F/K$.

**Proof.** Suppose that the polynomials $f_1, \ldots, f_r \in K[X_1, \ldots, X_n, Y_1, \ldots, Y_m]$ define $D_1$ and that the polynomials $g_1, \ldots, g_s \in K[X_1', \ldots, X_{n_2}', Z_1, \ldots, Z_l]$ define $D_2$. This time, we suppose that all the variables $X_i, X_i', Y_i, Z_i$ are distinct. Then the polynomials $f_1, \ldots, f_r, g_1, \ldots, g_s$ define $D_1 \times D_2$.

**Lemma 3.5.** Let $D$ be an $n$-dimensional diophantine family over $K$ and let $f = \left( \frac{g_1}{h_1}, \ldots, \frac{g_k}{h_k} \right)$ be a tuple of rational functions with $g_i, h_i \in K[X_1, \ldots, X_n]$ such that for every $i$ the polynomials $g_i$ and $h_i$ are coprime. Then there is an $k$-dimensional diophantine family $fD$ with

$$(fD)(F) = \left\{ \left( \frac{g_1(x)}{h_1(x)}, \ldots, \frac{g_k(x)}{h_k(x)} \right) \mid x \in D(F), h_i(x) \neq 0 \text{ for all } i \right\}$$

for every $F/K$.

**Proof.** Let $f_1, \ldots, f_r \in K[X_1, \ldots, X_n, Y_1, \ldots, Y_m]$ define $D$. Then a tuple $(z_1, \ldots, z_k) \in F^k$ is an element of the right hand side if and only if there exists $(x_1, \ldots, x_n, y_1, \ldots, y_m, w_1, \ldots, w_k) \in F^{n+m+k}$ such that

1. $g_i(x_1, \ldots, x_n) - h_i(x_1, \ldots, x_n) = 0$ for all $i = 1, \ldots, k$,
2. $w_ih_i(x_1, \ldots, x_n) = 1$ for all $i = 1, \ldots, k$, and
3. $f_j(x_1, \ldots, x_n, y_1, \ldots, y_m) = 0$ for all $j = 1, \ldots, r$.

Each of these conditions is the vanishing of a polynomial in the variables $W_1, \ldots, W_k, X_1, \ldots, X_k, Y_1, \ldots, Y_r$ and $Z_1, \ldots, Z_k$ over $K$.

**Remark 3.6.** Perhaps the most trivial 1-dimensional diophantine family over $K$ is the one assigning the set $F$ to every field $F/K$. As described above in Section 2.1, given a rational function $\gamma \in K(X)$ and a field $F/K$, we write $\gamma(F)$ to mean the image under
γ of F \{poles of γ\}. By this small abuse of notation, γ may be identified with the map which sends a field F/K to its image γ(F) under γ. Then by Lemma 3.5, γ is a 1-dimensional diophantine family over K. This applies in particular to the Kochen operator γ^n^\ast.

Lemma 3.7. If D is an n-dimensional diophantine family over K and a = (a_1, …, a_r) ∈ K^r, r < n, then there is an (n − r)-dimensional family D_a over K with

\[ D_a(F) = \{ x ∈ F^{n-r} \mid (x, a) ∈ D(F) \} \]

for every F/K.

Proof. Again, let f_1, …, f_r ∈ K[X_1, …, X_n, Y_1, …, Y_m] define D. We write

\[ g_i(X_1, …, X_{n-r}, Y_1, …, Y_m) := f_i(X_1, …, X_{n-r}, a_1, …, a_r, Y_1, …, Y_m). \]

Then the polynomials g_1, …, g_r ∈ K[X_1, …, X_{n-r}, Y_1, …, Y_m] define the (n − r)-dimensional diophantine family D_a over K.

Example 3.8. Each of the R^n_{\p,n} is a 1-dimensional diophantine family over K.

Proposition 3.9. Let D, D_1, D_2, … be n-dimensional diophantine families over K. If D(F) ⊆ \bigcup_{n∈\mathbb{N}} D_1(F) for every extension F/K, then there exists N such that D(F) ⊆ \bigcup_{i=1}^N D_i(F) for every extension F/K.

Proof. In light of Remark 3.2, this is a direct consequence of the compactness theorem of model theory, see for example [16, Thm. 2.1.4].

Proposition 3.10. Let D be a 1-dimensional diophantine family over K and let \mathcal{K} be a class of extensions of K. If

(i) D(L) = R^n_{\p}(L) for every L ∈ \mathcal{K}, and

(ii) D(E) ⊆ O_E for every finite extension E/K_p of relative type at most τ,

then there exists N such that π_n^\ast(L) ≤ N for every L ∈ \mathcal{K}.

Proof. Let F be any extension of K. For \p ∈ S_p^\ast(F) let (F', \p') denote a p-adic closure of (F, \p) (see [22, §3]). By the p-adic Lefschetz principle, the assumption (ii) implies that D(F') ⊆ O_{\p'}, in particular D(F) ⊆ O_{\p'} ∩ F = O_{\p}. (In model-theoretic terms, F' is elementarily equivalent, in the language of valued fields, to a finite extension E of K_p of relative type at most τ. More precisely, if F_0 denotes the algebraic part of F', then both F_0K_p and F' are elementary extensions of F_0 by [22, Thm. 5.1].) In particular, D(F) ⊆ \bigcap_{\p ∈ S_p^\ast(F)} O_{\p} = R^n_{\p}(F). So since R^n_{\p}(F) = \bigcup_{n=1}^\infty R^n_{p,n}(F), by Proposition 3.9 there exists N such that D(F) ⊆ \bigcup_{n=1}^N R^n_{p,n}(F) for every F/K. In fact (R^n_{p,n}(F))_{n∈\mathbb{N}} is an increasing chain, so D(F) ⊆ \bigcup_{n=1}^\infty R^n_{p,n}(F). Thus for L ∈ \mathcal{K}, (i) implies that R^n_{\p}(L) = D(L) ⊆ R^n_{p,N}(L), which shows that π_n^\ast(L) ≤ N.

Remark 3.11. We also have the following converse: If π_n^\ast(L) ≤ N for all L ∈ \mathcal{K}, then D = R^n_{p,N} is a diophantine family satisfying both conditions. This indicates that while our definition of π_n^\ast depends on the construction of the height function on polynomials over O_p, the property of a class \mathcal{K} to have bounded (p, τ)-Pythagoras number is a very robust notion and does not depend on the details of the height function.

Remark 3.12. The notion that a class \mathcal{K} has bounded (p, τ)-Pythagoras number is robust in a further sense: under taking a suitable alternative for the Kochen operator. Consider a rational function δ ∈ K(X) and suppose that R^n_{\delta}(F) is the integral closure in F of the ring

\[ R^\prime(F) := \left\{ \frac{a}{1 + t_p b} \mid a, b ∈ O_p[δ(F)], 1 + t_p b \neq 0 \right\}, \]

for every extension F/K. We introduce a new 1-dimensional diophantine family R^n_{\p,δ} over K, by defining R^n_{\p,δ}(F) in terms of δ exactly as R^n_{p,n}(F) is defined in terms of γ^n_\ast. Then

\[ R^n_{\p,δ}(F) = \bigcup_{n=1}^\infty R^n_{\delta}(F), \]
for all $F/K$. Simply adapting the proof of Proposition 3.10, a class $\mathcal{K}$ of extensions of $K$ has bounded $(p, \tau)$-Pythagoras number if and only if there is $M \in \mathbb{N}$ such that $R_{\mathcal{K}}^M(L) = R_{\mathcal{K}}^\tau(L)$, for all $L \in \mathcal{K}$. Also note that at least in the case $\tau = (1, 1)$, the Kochen-atom $\gamma^\tau_{p\nu}$ is universal in the sense that every such $\delta$ is in fact a rational function in $\gamma^\tau_{p\nu}$, see [22, Cor. 7.12].

4  | THE $(p, \tau)$-PYTHAGORAS NUMBER OF NUMBER FIELDS

Introduced by Poonen ([21]), and subsequently used and developed by others including Koenigsmann ([14]) and the second author ([7]), the following diophantine predicates behave well in local fields, and satisfy a strong local-global principle. They are defined from central simple algebras. For further details about central simple algebras, the Brauer group, and associated local-global principles, see [19, Sect. 6.3].

Let $A$ be a central simple algebra of prime degree $\ell$ over a field $F$. Following [7, Sect. 2], we let

$$S_A(F) := \{ \text{Trd}(x) \mid x \in A, \text{Nrd}(x) = 1 \} \subseteq F,$$

where Trd and Nrd are the reduced norm and reduced trace, see [12, Construction 2.6.1] for details. We also define

$$T_A(F) := \begin{cases} S_A(F) & \text{if } \ell > 2, \\ S_A(F) - S_A(F) & \text{if } \ell = 2. \end{cases}$$

If $A$ is a central simple algebra over $F$ and $E/F$ is any extension, we view $A_E := A \otimes_F E$ as a central simple algebra over $E$ and write $S_A(E) := S_{A_E}(E)$ and $T_A(E) := T_{A_E}(E)$.

**Lemma 4.1.** Both $S_A$ and $T_A$ are 1-dimensional diophantine families over $F$.

**Proof.** This is shown in [7, Lem. 2.12] and the subsequent discussion. $\square$

Recall that $A$ is split if it is isomorphic to a matrix algebra over $F$, and $A$ splits over $E$ if $A_E$ is split. The behaviour of $S_A$ and $T_A$ in a completion $F$ of a number field $L$ is determined by whether or not $A$ splits over $F$, and the behaviour of $S_A$ and $T_A$ in $L$ is controlled by a local-global principle, which leads to the following:

**Proposition 4.2** ([7, Prop. 2.9]). Let $L$ be a number field and $A$ a central simple algebra over $L$ of prime degree $\ell$ which splits over all real completions of $L$. Then

$$T_A(L) = \bigcap_p \mathcal{O}_p,$$

where the intersection is over the finitely many finite primes $p$ of $L$ such that $A$ does not split over $L_p$.

**Proposition 4.3** (see [7, Prop. 2.6]). Let $F$ be a non-archimedean local field of characteristic zero and let $A$ be a central simple algebra over $F$ of prime degree $\ell$. If $A$ is non-split then $T_A(F) = \mathcal{O}_F$.

Note that [7, Prop. 2.6] is stated for central division algebras of prime degree, but a non-split central simple algebra of prime degree is a division algebra.

Recall that above we fixed a number field $K$, a finite place $p$ of $K$, and a pair $\tau = (e, f) \in \mathbb{N}^2$. Given this data $(K, p, \tau)$, we now describe a choice of algebras $A, B$ over $K$.

**Proposition 4.4.** For every prime number $\ell$ there exist central simple algebras $A, B$ of degree $\ell$ over $K$ such that

1. neither of them splits over $K_p$,
2. for every finite place $q \neq p$ of $K$, at least one of them splits over $K_q$,
3. for every infinite place $q$ of $K$, both of them split over $K_q$.

**Proof.** The Brauer equivalence classes $[A]$ of central simple algebras $A$ over a field $F$ form the Brauer group $\text{Br}(F)$ of $F$, see [19, (6.3.2) Def.]. For an extension $F/K$, there is a group homomorphism $\text{Br}(K) \rightarrow \text{Br}(F)$ given by $[A] \mapsto [A_F]$. Moreover,
the local Hasse invariant is an isomorphism
\[
\text{inv}_{K_q} : \text{Br}(K_q) \rightarrow \begin{cases} 
\mathbb{Q}/\mathbb{Z} & \text{if } q \text{ is finite}, \\
\frac{1}{2}\mathbb{Z}/\mathbb{Z} & \text{if } q \text{ is infinite and } K_q \cong \mathbb{R}, \\
0 & \text{if } q \text{ is infinite and } K_q \cong \mathbb{C}, 
\end{cases}
\] (4.1)

and so \(A\) splits over \(K_q\) if and only if \(\text{inv}_{K_q}([A]) = 0\). There will be no ambiguity if we write \(\text{inv}_{K_q}([A]) = \text{inv}_{K_q}([A_{K_q}]\)). Note that each of the local Hasse invariants \(\text{inv}_{K_q}\) takes its values in \(\mathbb{Q}/\mathbb{Z}\).

The Albert–Brauer–Hasse–Noether Theorem ([19, (8.1.17) Thm.]) gives the exact sequence
\[
0 \rightarrow \text{Br}(K) \rightarrow \bigoplus_{q \in \mathcal{S}(K)} \text{Br}(K_q) \xrightarrow{\text{inv}_K} \mathbb{Q}/\mathbb{Z} \rightarrow 0,
\] (4.2)

where \(\mathcal{S}(K)\) is the set of (finite and infinite) places of \(K\), and \(\text{inv}_K\) is the sum of the local invariant maps \(\text{inv}_{K_q}\).

Fix two distinct finite places \(q_1, q_2 \neq p\) of \(K\). We define two sequences \((a_q)_{q \in \mathcal{S}(K)}\) and \((b_q)_{q \in \mathcal{S}(K)}\) of rational numbers, indexed by the places of \(K\), by
\begin{itemize}
  \item \(a_p = b_p = \ell^{-1}\),
  \item \(a_{q_1} = (\ell - 1)\ell^{-1}\) and \(b_{q_1} = 0\),
  \item \(a_{q_2} = 0\) and \(b_{q_2} = (\ell - 1)\ell^{-1}\),
  \item \(a_q = b_q = 0\), for every other place \(q\).
\end{itemize}

Note that only finitely many of the elements of these sequences are nonzero. Thus, by applying the inverses of the local Hasse invariants from (a), the sequences \((a_q)_{q \in \mathcal{S}(K)}\) and \((b_q)_{q \in \mathcal{S}(K)}\) correspond to elements of the direct sum \(\bigoplus_q \text{Br}(K_q)\). We also note the sums
\[
\sum_{q \in \mathcal{S}(K)} a_q = \sum_{q \in \mathcal{S}(K)} b_q = 0 \quad \text{in } \mathbb{Q}/\mathbb{Z}.
\]

By the exactness of the short exact sequence (4.2), we get (unique) equivalence classes \([A]\) and \([B]\) in \(\text{Br}(K)\) such that 
\(\text{inv}_{K_q}([A]) = a_q + \mathbb{Z}\) and \(\text{inv}_{K_q}([B]) = b_q + \mathbb{Z}\), for all \(q \in \mathcal{S}(K)\). Thus both \([A]\) and \([B]\) are of period \(\ell\). As \(K\) is a number field, this implies that they are also of index \(\ell\) ([23, 32.19]), which means that if \(A\) and \(B\) denote the unique division algebras in \([A]\) respectively \([B]\), then these are of degree \(\ell\).

**Proposition 4.5.** Let \(\ell\) be a prime number with \(\ell > ef\). If \(A\) and \(B\) are algebras as in Proposition 4.4, then

(i) for all finite extensions \(E/K_p\) of relative type at most \(r\),
\[
T_A(E) + T_B(E) = \mathcal{O}_E;
\]

(ii) and for all number fields \(L/K\),
\[
T_A(L) + T_B(L) \supseteq \bigcap_{q \in \mathcal{S}_p(L)} \mathcal{O}_q.
\]

**Proof.** First, suppose that \(E/K_p\) is a finite extension of relative type at most \(r\). Thus \([E : K_p] \leq ef < \ell\), so since \(A\) and \(B\) do not split over \(K_p\), they also do not split over \(E\) by [12, Cor. 4.5.9]. Therefore we may apply Proposition 4.3 to obtain
\[
T_A(E) + T_B(E) = \mathcal{O}_E + \mathcal{O}_E = \mathcal{O}_E.
\]

Next, let \(L/K\) be any number field and let \(\mathcal{Q}\) be a prime of \(L\) which lies over a prime \(q\) of \(K\). If \(q \neq p\), then at least one of \(A\) and \(B\) splits over \(K_q\) and therefore also over the completion \(L_{\mathcal{Q}}\) by construction. Hence
\[
T_A(L) + T_B(L) = \bigcap_{A \notin \mathcal{Q}} \mathcal{O}_A + \bigcap_{B \notin \mathcal{Q}} \mathcal{O}_B = \bigcap_{A \notin \mathcal{Q}} \mathcal{O}_A \supseteq \bigcap_{q \in \mathcal{S}_p(L)} \mathcal{O}_q.
\]
where the first equality is Proposition 4.2 and the second equality follows from weak approximation (see e.g. [9, 1.1.3]).

As before, fix a uniformizer \( t_p \in K \) of \( p \). For central simple algebras \( A, B \) over \( K \) and an extension \( F/K \) we define \( D^r_{p,t_p,A,B}(F) \) as

\[
\left\{ \frac{x}{1 + t_p w r \cdot y} \mid x, y \in T_A(F) + T_B(F), w \in \gamma^r_{p,t_p}(F), 1 + t_p w r \cdot y \neq 0 \right\}.
\]

**Lemma 4.6.** \( D^r_{p,t_p,A,B} \) is a 1-dimensional diophantine family over \( K \).

**Proof.** We have seen in Lemma 4.1 that \( T_A \) and \( T_B \) are 1-dimensional diophantine families over \( K \). The claim follows by applying Lemma 3.5 to the 5-dimensional diophantine family \( T_A \times T_B \times T_A \times T_B \times \gamma^r_{p,t_p} \) over \( K \) (Lemma 3.4) and the rational function \( (X_1 + X_2)(1 + t_p X_5^r(1 + X_3 + X_4))^{-1} \).

**Proposition 4.7.** If \( A, B \) are \( K \)-algebras as in Proposition 4.4, then

\[
D^r_{p,t_p,A,B}(E) \subseteq \mathcal{O}_E
\]

for every finite extension \( E/K \) of relative type at most \( r \).

**Proof.** By Proposition 4.5(i), we have \( T_A(E) + T_B(E) = \mathcal{O}_E \). Since also \( \gamma^r_{p,t_p}(E) \subseteq \mathcal{O}_E \) and \( 1 + t_p \mathcal{O}_E \subseteq \mathcal{O}_E^r \), we have \( D^r_{p,t_p,A,B}(E) \subseteq \mathcal{O}_E \), as required.

**Proposition 4.8.** If \( A, B \) are \( K \)-algebras as in Proposition 4.4, then

\[
D^r_{p,t_p,A,B}(L) = R^r(L)
\]

for every number field \( L \) containing \( K \).

**Proof.** By Proposition 4.7, \( D^r_{p,t_p,A,B}(L) \) \( \subseteq \mathcal{O}_{L_p} \) for every \( p \in S^r_p(L) \), hence

\[
D^r_{p,t_p,A,B}(L) \subseteq \bigcap_{p \in S^r_p(L)} \mathcal{O}_{L_p} \cap L = \bigcap_{p \in S^r_p(L)} \mathcal{O}_p = R^r(L).
\]

To show the other inclusion, let \( r \in R^r_p(L) \). Since \( L/K \) is finite, the set \( S^r_p(L) \) of primes of \( L \) over \( p \) is finite. Write \( \mathcal{P}_1, \ldots, \mathcal{P}_k \in S^r_p(L) \) for the primes over \( p \) of relative type \( \leq r \), and \( \mathcal{Q}_1, \ldots, \mathcal{Q}_l \) for the primes over \( p \) not of relative type \( \leq r \). For each \( i \in \{1, \ldots, l\} \), by Lemma 2.12 there exists \( z_i \) such that

\[
v_{\mathcal{Q}_l} \left( \gamma^r_{p,t_p}(z_i) \right) \leq -\frac{1}{e + 1} v_{\mathcal{Q}_l}(t_p),
\]

i.e. \( v_{\mathcal{Q}_l} \left( \left( t_p \gamma^r_{p,t_p}(z_i) \right)^{r+1} \right)^{-1} \geq 0 \). By weak approximation and continuity of rational functions, there exists \( z \in L \) such that

\[
v_{\mathcal{Q}_l} \left( \left( t_p \gamma^r_{p,t_p}(z) \right)^{r+1} \right)^{-1} \geq 0 \quad \text{for each } i \in \{1, \ldots, l\}.
\]

By another application of weak approximation there exists \( y \in L \) such that

\[
v_{\mathcal{Q}_l} \left( \left( t_p \gamma^r_{p,t_p}(z) \right)^{r+1} \right)^{-1} + y \geq \max \left\{ 0, -v_{\mathcal{Q}_l} \left( \left( t_p \gamma^r_{p,t_p}(y) \right)^{r+1} \right) \right\}, \quad i = 1, \ldots, l,
\]

\[
v_{\mathcal{Q}_l}(y) \geq 0, \quad i = 1, \ldots, k.
\]

In particular, \( y \in \bigcap_{p \in S^r_p(L)} \mathcal{O}_p \) and \( x := r \left( 1 + t_p \gamma^r_{p,t_p}(z)^{r+1} \right) \) satisfies \( v_{\mathcal{Q}_l}(x) \geq 0 \) for each \( i \in \{1, \ldots, l\} \). As \( \mathcal{P}_l \in S^r_p(L) \), we have \( r, t_p, \gamma^r_{p,t_p}(z), y \in \mathcal{O}_{\mathcal{P}_l} \), hence \( v_{\mathcal{P}_l}(x) \geq 0 \) for all \( i \in \{1, \ldots, k\} \). Thus \( x \in \bigcap_{p \in S^r_p(L)} \mathcal{O}_p \). As

\[
\bigcap_{p \in S^r_p(L)} \mathcal{O}_p \subseteq T_A(L) + T_B(L)
\]
by Proposition 4.5(ii), we get that
\[ r = x \left( 1 + t_p \gamma_p (z) \right) \in D_{p,A,B}(L), \]
as required.

**Theorem 4.9.** For every finite place \( p \) of a number field \( K \) and every \( \tau \in \mathbb{N}_2 \), there exists \( N \in \mathbb{N} \) such that \( \pi_p^\tau(L) \leq N \) for every number field \( L \) containing \( K \).

**Proof.** We choose algebras \( A \) and \( B \) over \( K \) according to Proposition 4.4, and we apply Proposition 3.10 to the class \( \mathcal{K} \) of finite extensions \( L/K \) and the diophantine family \( D = D_{p,A,B}^\tau \), where the two assumptions of Proposition 3.10 are verified in Proposition 4.8 and Proposition 4.7, respectively.

**Remark 4.10.** Given an arbitrary field \( F \supseteq K \) there is no obvious relation between \( \pi_p^\tau(F) \) and \( \pi_p^\tau'(F) \) for \( \tau \neq \tau' \). For example if \( \tau \leq \tau' \) then we have \( R_p^\tau(F) \supseteq R_p^\tau'(F) \), but also \( \gamma_p^\tau \neq \gamma_p^\tau' \). Likewise, there is no reason to suspect that the bounds \( N \) in Theorem 4.9 should be related for different choices of \( \tau \).

### 5 | THE \((p, \tau)\)-PYTHAGORAS NUMBER IN FINITE EXTENSIONS

The growth of the classical Pythagoras number is bounded in finite extensions \( E/F \) by
\[ \pi(E) \leq [E : F] \cdot \pi(F), \]
see [20, Ch. 7, Prop. 1.13]. We now combine ideas from the proof of Theorem 4.9 with techniques for \( p \)-valuations on general fields to prove an (imexpicit) analogue of this for the \((p, \tau)\)-Pythagoras number.

As before fix \( K, p \) and \( \tau = (e, f) \) and let \( F/K \) be an extension. We equip \( S_p^\tau(F) \) with the constructible topology, which by definition has a basis consisting of the sets
\[ S_p^\tau(F; a) := \{ \mathfrak{P} \in S_p^\tau(F) \mid v_{\mathfrak{P}}(a) \geq 0 \}, \quad a \in F, \]
and their complements. In [1], we studied approximation theorems for spaces of localities, i.e. valuations, orderings, and absolute values, on a given field. We now deduce an approximation theorem in the setting of the space \( S_p^\tau(F) \).

**Theorem 5.1.** Let \( S_1, \ldots, S_n \subseteq S_p^\tau(F) \) be disjoint and closed, let \( x_1, \ldots, x_n \in F \), and let \( z_1, \ldots, z_n \in F^\times \). Assume that, for any \( \mathfrak{P} \in S_i \) and \( \mathfrak{P} \in S_j \), if the valuation \( w \) is the finest common coarsening of \( v_{\mathfrak{P}} \) and \( v_{\mathfrak{P}} \), then \( w(x_i - x_j) \geq w(z_i) = w(z_j) \). Then there exists \( x \in F \) with
\[ v_{\mathfrak{Q}}(x - x_i) > v_{\mathfrak{Q}}(z_i) \quad \text{for all} \quad \mathfrak{Q} \in S_i, \quad \text{for} \quad i = 1, \ldots, n. \]

**Proof.** Corollary 5.5 of [1] is a similar statement in which \( S_p^\tau(F) \) is replaced by a space \( S_p^\tau(F) \), for \( \pi \in F^\times \) and \( e \in \mathbb{N} \). By definition (see [1, Example 2.4]), \( S_p^\tau(F) \) is the space of equivalence classes of valuations \( v \) on \( F \) with value group \( \Gamma_v \), which has \( \mathbb{Z} \) as a convex subgroup and \( 0 < v(\pi) \leq e \). We note that \( S_p^\tau(F) \subseteq S_p^\tau(F) \), and if we equip \( S_p^\tau(F) \) with its own constructible topology (see [1, Sect. 2]) then \( S_p^\tau(F) \) is a closed subset: By [22, Lem. 6.2], \( S_p^\tau(F) \) is the intersection over all sets \( \{ v \in S_p^\tau(F) : v(a) \geq 0 \} \) for \( a \in \mathfrak{O}_p \cup \gamma_p^\tau(F) \). Therefore, each \( S_i \) is also a closed subset of \( S_p^\tau(F) \) and so we may obtain the required element \( x \in F \) by an application of [1, Cor. 5.5].

**Lemma 5.2.** Let \( \tau \leq \tau' \in \mathbb{N}_2 \). There is a rational function \( \omega_{\tau, \tau'} \in \mathbb{Q}(t_p)(X) \) such that \( v_{\mathfrak{P}}(\omega_{\tau, \tau'}(x)) > 0 \) for all \( x \in F \) and \( \mathfrak{P} \in S_p^\tau(F) \), and moreover \( v_{\mathfrak{P}}(\omega_{\tau, \tau'}(x)) = 1 \) if \( v_{\mathfrak{P}}(x) = 1 \) and \( \mathfrak{P} \) is of exact relative type \( \tau \) over \( p \).

**Proof.** Write \( \tau' = (e', f') \). By Dirichlet’s theorem on primes in arithmetic progressions there exists \( k \in \mathbb{N} \) such that \( \ell' := 1 + ke' \) is a prime number and \( \ell' > e' \). Let \( \beta(X) = t_p^{-k} X^{f'} \). For every \( \mathfrak{P} \in S_p^\tau(F) \) and \( x \in F \) we have \( v_{\mathfrak{P}}(\beta(x)) = \ell v_{\mathfrak{P}}(x) - k v_{\mathfrak{P}}(t_p) \), which is non-zero (since \( \ell' > k \) and \( \ell' \geq v_{\mathfrak{P}}(t_p) \) imply \( \ell' \neq k v_{\mathfrak{P}}(t_p) \)), and equals 1 if \( v_{\mathfrak{P}}(x) = 1 \) and \( v_{\mathfrak{P}}(t_p) = e \). Thus \( \omega_{\tau, \tau'}(X) = (\beta(X) + \beta(X)^{-1})^{-1} \) satisfies the claim.
Lemma 5.3. There is a rational function $\rho_\tau \in \mathbb{Q}(X)$ such that for all $\mathfrak{P} \in S_{p}^\tau(F)$ and all $x \in F$ we have

$$v_\mathfrak{P}(\rho_\tau(x)) \begin{cases} = 0, & \text{if } v_\mathfrak{P}(x) = 0, \\ > 0, & \text{if } v_\mathfrak{P}(x) \neq 0, \end{cases}$$

and if $v_\mathfrak{P}(x) = 0$ then $\text{res}_\mathfrak{P}(\rho_\tau(x)) = \text{res}_\mathfrak{P}(x)$.

Proof. Write $\rho_\tau(X) = X \left( X^{q^\ell} - X + 1 \right)^{-1}$. Let $\mathfrak{P} \in S_{p}^\tau(F)$ and let $x \in F$. If $v_\mathfrak{P}(x) < 0$ then $v_\mathfrak{P}(x^{q^\ell} - x + 1) = q^\ell v_\mathfrak{P}(x) < 0$, and so $v_\mathfrak{P}(\rho_\tau(x)) = (1 - q^\ell) v_\mathfrak{P}(x) > 0$. On the other hand, if $v_\mathfrak{P}(x) > 0$ then $v_\mathfrak{P}(x^{q^\ell} - x + 1) = 0$, so $v_\mathfrak{P}(\rho_\tau(x)) = v_\mathfrak{P}(x) > 0$. Finally, if $v_\mathfrak{P}(x) = 0$ then

$$\text{res}_\mathfrak{P}(x^{q^\ell} - x + 1) = \text{res}_\mathfrak{P}(x)^{q^\ell} - \text{res}_\mathfrak{P}(x) + 1 = 1 \neq 0,$$

and in particular $v_\mathfrak{P}(x^{q^\ell} - x + 1) = 0$. Therefore $v_\mathfrak{P}(\rho_\tau(x)) = 0$ and $\text{res}_\mathfrak{P}(\rho_\tau(x)) = \text{res}_\mathfrak{P}(x)$. $\square$

Proposition 5.4. Let $\tau \leq \tau' = (\epsilon', f')$ and let $S_0$ denote an open-closed subset of $S_{p}^{\tau'}(F)$ such that $S_{p}^\tau(F) \subseteq S_0$. There exists $y \in F$ such that

$$v_\mathfrak{P}\left(\gamma_\mathfrak{P}^{\tau'}(y)\right) \begin{cases} \in [0, e'q^f], & \text{if } \mathfrak{P} \in S_0, \\ < 0, & \text{if } \mathfrak{P} \in S_{p}^{\tau'}(F) \setminus S_0. \end{cases}$$

Proof. For each $\mathfrak{P} \in S_{p}^{\tau'}(F) \setminus S_0$, we choose $y_\mathfrak{P} \in F$ as follows. First, if the relative type of $\mathfrak{P}$ is exactly $\tau'' = (e'', f'')$ with $e'' > e$, then let $t_\mathfrak{P}$ be a uniformizer of $v_\mathfrak{P}$ and set $y_\mathfrak{P} = \omega_{e'', t_\mathfrak{P}}(t_\mathfrak{P})$. By Lemma 5.2, $v_\mathfrak{P}(y_\mathfrak{P}) = 1$; and by Lemma 2.12, $v_\mathfrak{P}\left(\gamma_\mathfrak{P}^{\tau'}(y_\mathfrak{P})\right) < 0$. Also, for all $\mathfrak{Q} \in S_{p}^{\tau'}(F)$ we have $v_\mathfrak{Q}(y_\mathfrak{P}) < 0$. In particular, $y_\mathfrak{P} \in R_{p}^{\tau'}(F)$.

On the other hand, if the relative type of $\mathfrak{P}$ is exactly $\tau'' = (e'', f'')$ with $f'' < f$, then let $a_\mathfrak{P}$ with $v_\mathfrak{P}(a_\mathfrak{P}) = 0$ and $\text{res}_\mathfrak{P}(a_\mathfrak{P})$ a generator of $Fv_\mathfrak{Q}$, and set $y_\mathfrak{P} = \rho_\tau(a_\mathfrak{P})$. By Lemma 5.3, $v_\mathfrak{P}(y_\mathfrak{P}) = 0$ and $\text{res}_\mathfrak{P}(y_\mathfrak{P})$ is a generator of $Fv_\mathfrak{Q}$. By Lemma 2.12, we have $v_\mathfrak{P}\left(\gamma_\mathfrak{P}^{\tau'}(y_\mathfrak{P})\right) < 0$. Also, for all $\mathfrak{Q} \in S_{p}^{\tau'}(F)$ we have $v_\mathfrak{Q}(y_\mathfrak{P}) \geq 0$, i.e. $y_\mathfrak{P} \in R_{p}^{\tau'}(F)$.

In either case, we have chosen $y_\mathfrak{P} \in R_{p}^{\tau'}(F)$ such that $v_\mathfrak{P}\left(\gamma_\mathfrak{P}^{\tau'}(y_\mathfrak{P})\right) < 0$. Next we make use of the compactness of $S_{p}^{\tau'}(F)$. For $y \in F$, we let

$$S_y = \left\{ \mathfrak{P} \in S_{p}^{\tau'}(F) \mid v_\mathfrak{P}\left(\gamma_\mathfrak{P}^{\tau'}(y)\right) < 0 \right\}.$$

Each $S_y$ is an open-closed subset of $S_{p}^{\tau'}(F)$. By our choice of the elements $y_\mathfrak{P}$, the family

$$\left\{ S_{y_\mathfrak{P}} \setminus S_0 : \mathfrak{P} \in S_{p}^{\tau'}(F) \setminus S_0 \right\}$$

is an open covering of $S_{p}^{\tau'}(F) \setminus S_0$. So by compactness there exist $\mathfrak{P}_1, \ldots, \mathfrak{P}_n \in S_{p}^{\tau'}(F) \setminus S_0$ such that with $S_i' := S_{y_{\mathfrak{P}_i}}$, we have

$$S_{p}^{\tau'}(F) = S_0 \cup S_1' \cup \cdots \cup S_n'. $$

Choose open-closed sets $S_1 \subseteq S_1', \ldots, S_n \subseteq S_n'$ such that

$$S_{p}^{\tau'}(F) = S_0 \cup S_1 \cup \cdots \cup S_n$$

is a partition. We seek to apply Theorem 5.1 to the sets $S_0, S_1, \ldots, S_n$, the elements $x_0 = t_0^{-1}, x_1 = y_\mathfrak{P}_1, \ldots, x_n = y_\mathfrak{P}_n$, and $z_0 = t_\mathfrak{P}, \ldots, z_n = t_\mathfrak{P}$. To verify that the hypothesis of the theorem holds, we argue as follows: let $w$ be any valuation on $F$ that is a common coarsening of valuations $v_\mathfrak{P}$ and $v_\mathfrak{Q}$ corresponding to primes $\mathfrak{P} \in S_i$ and $\mathfrak{Q} \in S_j$, for $i \neq j$. Note that $w$ is a proper coarsening of these valuations since $S_i$ and $S_j$ are disjoint and $v_\mathfrak{P}, v_\mathfrak{Q}$ are incomparable. Then $w(z_i) = w(z_j) = 0$ and $w(x_i - x_j) \geq 0$. Therefore, by Theorem 5.1, there exists $y \in F$ such that

$$v_\mathfrak{P}(y - x_i) > v_\mathfrak{P}(t_\mathfrak{P}),$$
for each \( \mathfrak{P} \in S \) and each \( i \). In particular, for \( \mathfrak{P} \in S_0 \) we have that \( v_{\mathfrak{P}}(y) = -v_{\mathfrak{P}}(t_p) < 0 \), hence
\[
v_{\mathfrak{P}}(\gamma_{p,t_p}^\tau(y)) = eq' v_{\mathfrak{P}}(t_p) - v_{\mathfrak{P}}(t_p) = (eq' - 1)v_{\mathfrak{P}}(t_p) \in \{0, \ldots, e'eq' \},
\]
cf. Lemma 2.11. On the other hand, for \( \mathfrak{Q} \in S_1 \), with \( i > 0 \), we get that \( v_{\mathfrak{Q}}(y - y_{\mathfrak{Q}}) > v_{\mathfrak{Q}}(t_p) \). Since we have
\[
v_{\mathfrak{Q}}(\gamma_{p,t_p}^\tau(y_{\mathfrak{Q}})) < 0, \quad v_{\mathfrak{Q}}(\gamma_{p,t_p}^\tau(y)) < 0 \text{ by Lemma 2.13.}
\]

Fix \( n, m \in \mathbb{N} \) and let \( e' = (e', f') \), where \( e' = me \) and \( f' = m_1f \). Let \( E \) be the class of fields \( E \) which contain some \( F/K \) with \([E : F] = m \) and \( \pi_p^\tau(F) = n \). We adapt the arguments of Section 4 in order to show that \( \pi_p^\tau(E) \) is bounded by a function of \( m, n \).

We let
\[D_{r,1}(F) := \{x \in F \mid \exists a_0, \ldots, a_{m-1} \in R_{p,n}^\tau(F) : x^m + a_{m-1}x^{m-1} + \cdots + a_0 = 0\},
\]
and
\[D_{r,2}(F) := \left\{\frac{a}{1 + t_p\gamma_{p,t_p}^\tau(y)^{e'}} b \mid a, b \in D_{r,1}(F), y \in F, \gamma_{p,t_p}^\tau(y) \neq \infty, 1 + t_p\gamma_{p,t_p}^\tau(y)^{e'} b \neq 0\right\}.
\]

**Lemma 5.5.** Both \( D_{r,1}(F) \) and \( D_{r,2}(F) \) are 1-dimensional diophantine families over \( K \).

**Proof.** This is very similar to Lemma 4.6. This time we use the fact that \( R_{p,n}^\tau \) is a 1-dimensional diophantine family over \( K \), as seen in Example 3.8. From this is immediately follows that \( D_{r,1}(F) \) is a 1-dimensional diophantine family over \( K \). To see that \( D_{r,2}(F) \) is a 1-dimensional diophantine family over \( K \) we now apply Lemma 3.5 to the 3-dimensional diophantine family \( D_{r,1}(F) \times D_{r,2}(F) \times \gamma_{p,t_p}^\tau \) and the rational function \( X_1(1 + t_pX_2X_3)^{-1} \).

**Proposition 5.6.** For every \( E \subseteq K \) we have \( D_{r,2}(F) \subseteq R_p^\tau(E) \).

**Proof.** Since \( R_p^\tau(E) \) is integrally closed in \( E \) and \( R_{p,n}^\tau(F) \subseteq R_p^\tau(E) \), we have \( D_{r,1}(F) \subseteq R_p^\tau(E) \). Let \( \mathfrak{P} \in S_p^\tau \). Then \( v_{\mathfrak{P}}(t_p) > 0 \). Furthermore, for \( y \in E \) and \( b \in R_{p,n}^\tau \), we have \( v_{\mathfrak{P}}(\gamma_{p,t_p}^\tau(y)^{e'} b) \geq 0 \), hence \( v_{\mathfrak{P}}(1 + t_p\gamma_{p,t_p}^\tau(y)^{e'} b) = 0 \). Therefore elements of the form \( a(1 + t_p\gamma_{p,t_p}^\tau(y)^{e'} b)^{-1} \) are contained in \( R_p^\tau(E) \), where \( a, b \in D_{r,1}(F) \) and \( y \in E \). This establishes \( D_{r,2}(F) \subseteq R_p^\tau(E) \).

**Lemma 5.7.** For every \( E \subseteq E \) we have \( R_{p,n}^\tau(E) \subseteq D_{r,1}(F) \).

**Proof.** Choose \( F \) such that \([E : F] = m \) and \( \pi_p^\tau(F) = n \), although the choice of \( F \) will not matter. Let \( S \) be the set of primes of \( E \) (of arbitrary type) lying over elements of \( S_p^\tau \). By our choice of \( \tau' \), we have \( S \subseteq S_p^\tau \). If we denote by \( A \) the integral closure of \( R_p^\tau(F) \) in \( E \), then \( A \) is the holomorphy ring corresponding to \( S \) and we have
\[R_p^\tau(E) \subseteq A \subseteq R_p^\tau(E).
\]
Since \( \pi_p^\tau(F) = n \), we have \( R_p^\tau(F) = R_{p,n}^\tau(F) \); and trivially \( R_p^\tau(F) \subseteq R_{p,n}^\tau(F) \). As the degree of the extension \( E/F \) is \( m \), \( D_{r,1}(F) \) contains the integral closure of \( R_p^\tau(F) \) in \( E \), which is \( A \). In particular \( R_{p,n}^\tau(E) \subseteq D_{r,1}(F) \).

**Proposition 5.8.** For every \( E \subseteq E \) we have \( D_{r,2}(F) \subseteq R_p^\tau(E) \).

**Proof.** In view of Proposition 5.6, it only remains to show that \( R_p^\tau(E) \subseteq D_{r,2}(F) \). Let \( x \in R_p^\tau(E) \). In fact, we aim to find \( b \in R_p^\tau(E) \) and \( y \in E \) with
\[x(1 + t_p\gamma_{p,t_p}^\tau(y)^{e'} b) \in R_p^\tau(E),
\]
which we will do by applying Theorem 5.1. As \( R_p^\tau(E) \subseteq D_{p,m,n}^{\tau(1)} \) by Lemma 5.7, this will show that \( x \in D_{p,m,n}^{\tau(2)}(E) \). We define the sets
\[
S_0 := \{ \mathfrak{P} \in S_p^\tau(E) \mid v_\mathfrak{P}(x) \geq 0 \}
\]
and
\[
S_1 := S_p^\tau(E) \setminus S_0.
\]
Note that \( S_0 \) and \( S_1 \) are open-closed in \( S_p^\tau(E) \) and \( S_1 \cap S_p^\tau(E) = \emptyset \). We find a suitable element \( y \in E \) by a direct application of Proposition 5.4: we obtain \( y \in E \) such that
\[
v_\mathfrak{P}(\gamma_{\mathfrak{P},p}(y)) \begin{cases} \in [0,e'eq^\ell], & \text{if } \mathfrak{P} \in S_0, \\ < 0, & \text{if } \mathfrak{P} \in S_1. \end{cases}
\]
We obtain a suitable \( b \in E \) by solving a more straightforward approximation problem: By Theorem 5.1, there exists \( b \in R_p^\tau(E) \) such that
\[
v_\mathfrak{P}(b) \geq 0, \quad \text{if } \mathfrak{P} \in S_0,
\]
and
\[
v_\mathfrak{P} \left(b + t_p^{-1} \gamma_{\mathfrak{P},p}(y)^{e'} \right) \geq v_\mathfrak{P} \left(x^{-1} t_p^{-1} \gamma_{\mathfrak{P},p}(y)^{e'} \right), \quad \text{if } \mathfrak{P} \in S_1,
\]
Indeed, if a valuation \( u \) on \( E \) coarsens \( v_{\mathfrak{P}} \) and \( v_\Sigma \) for \( \mathfrak{P} \in S_0 \) and \( \Sigma \in S_1 \), \( v_{\mathfrak{P}}(x) \geq 0 \) and \( v_\Sigma(x) < 0 \) imply that \( u(x) = 0 \), and \( v_{\mathfrak{P}}(\gamma_{\mathfrak{P},p}(y)) \in [0,e'eq^\ell] \) implies that \( u(\gamma_{\mathfrak{P},p}(y)) = 0 \). Therefore also \( u \left(t_p \gamma_{\mathfrak{P},p}(y)^{e'} \right) = 0 \) and \( u \left(x t_p \gamma_{\mathfrak{P},p}(y)^{e'} \right) = 0 \). In particular, the hypothesis of the theorem is satisfied, and the \( b \in E \) so obtained lies in \( R_p^\tau(E) \).

For \( \mathfrak{P} \in S_0 \), we have \( v_{\mathfrak{P}}(t_p^{-1} \gamma_{\mathfrak{P},p}(y)^{e'}) < 0 \), hence
\[
v_{\mathfrak{P}} \left(b + t_p^{-1} \gamma_{\mathfrak{P},p}(y)^{e'} \right) = v_{\mathfrak{P}} \left(t_p^{-1} \gamma_{\mathfrak{P},p}(y)^{e'} \right), \quad \text{if } \mathfrak{P} \in S_0,
\]
\[
\geq v_{\mathfrak{P}} \left(x^{-1} t_p^{-1} \gamma_{\mathfrak{P},p}(y)^{e'} \right), \quad \text{if } \mathfrak{P} \in S_1,
\]
i.e.
\[
v_{\mathfrak{P}} \left(1 + t_p \gamma_{\mathfrak{P},p}(y)^{e'} b \right) = 0, \quad \text{if } \mathfrak{P} \in S_0,
\]
\[
v_{\mathfrak{P}} \left(x \left(1 + t_p \gamma_{\mathfrak{P},p}(y)^{e'} b \right) \right) \geq 0, \quad \text{if } \mathfrak{P} \in S_1.
\]
Since \( v_{\mathfrak{P}}(x) \geq 0 \) for \( \mathfrak{P} \in S_0 \), we obtain that \( x \left(1 + t_p \gamma_{\mathfrak{P},p}(y)^{e'} b \right) \in R_p^\tau(E) \).

**Theorem 5.9.** There is a function \( a_p^\tau : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \) such that
\[
\pi_p^\tau(E) \leq a_p^\tau(\pi_p^\tau(F), [E : F]),
\]
for every field extension \( E/F \) with \( \pi_p^\tau(F) < \infty \).

**Proof.** Let \( m, n \in \mathbb{N} \). We apply Proposition 3.10 to the class \( E \) and the diophantine family \( D_{p,m,n}^{\tau(2)} \), where the two assumptions of Proposition 3.10 are verified in Proposition 5.8 and Proposition 5.6, respectively. Thus there exists \( N \) such that \( \pi_p^\tau(E) \leq N \) for every \( E \in \mathcal{E} \), so we can choose \( a_p^\tau(n,m) = N \).

**Remark 5.10.** Beyond the statement of the theorem, we are unable to say much about the behaviour of the \((p, \tau)\)-Pythagoras number in finite extensions:

For example, it is known that the classical Pythagoras does not increase in finite extensions of number fields, cf. [20, Ch. 7, Example 1.4 (2) and (3)], but we don’t expect this to happen for the \((p, \tau)\)-Pythagoras number.

In fact, it is known that there are finite extensions of infinite algebraic extensions of \( \mathbb{Q} \) in which the classical Pythagoras number increases, see for instance [5, Example on p. 432], and one may expect that similar examples exist for the \((p, \tau)\)-Pythagoras number. For example, if \( F \) is the closure of \( \mathbb{Q} \) under adjoining preimages of \( \gamma_p \), one trivially has \( R_p(F) = F = \gamma_p(F) \), hence...
By definition, in any field $F$ with finite $(p, r)$-Pythagoras number the holomorphy ring $R_p^r(F)$ is a diophantine subset. In this section we generalize this observation, by showing in Corollary 6.5 that the same applies to the holomorphy rings associated to arbitrary open-closed subsets of $S_p^r(F)$. Theorem 6.4 is a uniform version of this fact.

As a technical tool, it turns out to be useful to extend some of the ideas from diophantine families over fields to commutative algebras which are finite-dimensional vector spaces over fields. To this end, we introduce a small piece of notation. Write $\pi_p(\mathcal{E}) = 1$. One can then deduce from a theorem of Weissauer [26, Satz 9.7] that in any proper finite extension $E$ of $F$ one has $R_p(\mathcal{E}) \neq \mathcal{E}$, and one might suspect that in fact $\pi_p(\mathcal{E}) > 1$, although this seems not easy to prove.

6 | DIOPHANTINE HOLOMORPHY RINGS OF $p$-VALUATIONS

By definition, in any field $F$ with finite $(p, r)$-Pythagoras number the holomorphy ring $R_p^r(F)$ is a diophantine subset. In this section we generalize this observation, by showing in Corollary 6.5 that the same applies to the holomorphy rings associated to arbitrary open-closed subsets of $S_p^r(F)$. Theorem 6.4 is a uniform version of this fact.

As a technical tool, it turns out to be useful to extend some of the ideas from diophantine families over fields to commutative algebras which are finite-dimensional vector spaces over fields. To this end, we introduce a small piece of notation. Write $X = (X_1, \ldots, X_n)$ and $Y = (Y_1, \ldots, Y_m)$. For $f_1, \ldots, f_r \in K[X, Y]$ and for any commutative (unital, associative) $F$-algebra $B$, we write

$$P_{f_1, \ldots, f_r}(B) := \{ x \in B^n \mid \exists y \in B^m : f_1(x, y) = \cdots = f_r(x, y) = 0 \}.$$ 

The following lemma is straightforward, but we include it for lack of a suitable reference.

**Lemma 6.1.** Let $f_1, \ldots, f_r \in K[X, Y]$ and let $l \in \mathbb{N}$. Then

$$F^n \cap P_{f_1, \ldots, f_r}(B) = \bigcap_{m \in \text{MaxSpec}(B)} (F^n \cap P_{f_1, \ldots, f_r}(B/m)),$$

for all extensions $F/K$, and all commutative $F$-algebras $B$ of dimension at most $l$. Here $F$ is identified with its image in $B$ and $B/m$.

**Proof.** Let $B$ be a commutative $F$-algebra which has dimension at most $l$ as an $F$-vector space. As $B$ is finite dimensional, it is Artinian, hence the Jacobson radical $\mathfrak{J}$ of $B$ is nilpotent ([3, Prop. 8.4]), and therefore more precisely $\mathfrak{J}^l = 0$. Then for all $s \in \{1, \ldots, r\}$, all extensions $F/K$, all $a \in F$, $x \in F^n$, and $y \in B^m$, we have

$$f_s(x, y)^l = 0 \iff f_s(x, y + j) = 0 \iff f_s(x, y + m) = 0, \text{ for all } m \in \text{MaxSpec}(B).$$

The result now follows from the Chinese Remainder Theorem. 

**Lemma 6.2.** Let $f_1, \ldots, f_r \in K[X, Y]$ and let $k \in \mathbb{N}$. There exists an $(n + k)$-dimensional diophantine family $D$ over $K$ such that

$$D(F) = \left\{ (x, z) \in F^n \times F^k \mid x \in P_{f_1, \ldots, f_r}(B_2) \right\},$$

for all extensions $F/K$, and where $B_2$ denotes the commutative $F$-algebra

$$F[T]/\left( T^k + \sum_{i=0}^{k-1} z_i T^i \right).$$

**Proof.** In a more advanced way, this construction can be described through the Weil restriction of the affine variety cut out by the polynomials $f_1, \ldots, f_r$, along the family of schemes described by the $B_2$, fibred over the parameter space $A^k$. Alternatively, from a model-theoretic standpoint, one can prove the statement by a quantifier-free interpretation of $B_2$ in $F$, uniformly in the parameter tuple $z$. We give an elementary description instead.

We introduce two new tuples of variables $Z = (Z_i)_{0 \leq i < k}$ and $U = (U_{ij})_{0 \leq i < k, 1 \leq j \leq m}$. We write

$$g(Z, T) := T^k + \sum_{i=0}^{k-1} Z_i T^i \in K[Z, T].$$
and, for each \( s \in \{1, \ldots, r\} \), we let

\[
\tilde{f}_s(X, U, T) := f_s \left( X, \sum_{i=0}^{k-1} U_{i,1} T^i, \ldots, \sum_{i=0}^{k-1} U_{i,m} T^i \right).
\]

Choose \( d \in \mathbb{N} \) to be the maximum of the degrees of the polynomials \( \tilde{f}_s \) in the variable \( T \), and introduce a new tuple of variables \( W = (W_i)_{0 \leq l \leq d} \). Then, for each \( s \), we consider the polynomial

\[
\tilde{f}_s(X, Z, U, W, T) := \tilde{f}_s(X, U, T) - g(Z, T) \sum_{l=0}^{d} W_l T^l.
\]

Note that \( \tilde{f}_s(x, z, u, w, T) = 0 \) for some \( w \) if and only if \( g(z, T) \) divides \( \tilde{f}_s(x, u, T) \) in \( F[T] \). By taking coefficients with respect to the variable \( T \), we obtain a family of polynomials \( h_{s,l} \in K[X, Z, U, W] \), for \( 1 \leq s \leq r \) and \( 0 \leq l \leq d + k \), such that

\[
\tilde{f}_s(X, Z, U, W, T) = \sum_{l=0}^{d+k} h_{s,l}(X, Z, U, W) T^l.
\]

We may define the required \((n + k)\)-dimensional diophantine family \( D \) over \( K \) by writing

\[
D(F) = \{(x, z) \in F^n \times F^k \mid \exists u \in F^{k+1}, w \in F^{d+1} : h_{s,l}(x, z, u, w) = 0 \text{ for all } s, l\},
\]

for \( F/K \).

**Lemma 6.3.** For every field extension \( F/K \) and every \( a \in F \), we have

\[
S_p^r(F; a) = \bigcup_{m \in \text{MaxSpec}(B_a)} \text{res}_{(B_a/m)/F}(S_p^r(B_a/m)),
\]

where \( \text{res}_{E/F} \) denotes restriction of primes from \( E \) to \( F \), and \( B_a \) is the commutative \( F \)-algebra

\[
F[T]/\left( t_p a^e \left( (T^{q'^l} - T)^2 - 1 \right) - (T^{q'^l} - T) \right).
\]

**Proof.** Denote \( \text{MaxSpec}(B_a) = \{m_1, \ldots, m_r\} \) and \( E_i = B_a/m_i \). Let

\[
g_a = t_p a^e \left( (T^{q'^l} - T)^2 - 1 \right) - (T^{q'^l} - T) \in F[T]
\]

and note that \( g_a \) is closely related to \( \gamma_{p,j,a}^r \).

First let \( \mathfrak{P} \in S_p^r(E_i) \) for some \( i \). If \( \theta \) denotes the residue of \( T \) in \( E_i \), we have \( \gamma_{p,j,a}^r(\theta) \in \mathcal{O}_{\mathfrak{P}} \) and therefore \( v_{\mathfrak{P}}(\theta^{q'^l} - \theta) > v_{\mathfrak{P}}\left( \theta^{q'^l} - \theta \right)^2 - 1 \). so we necessarily have \( v_{\mathfrak{P}}(t_p a)^e > 0 \) and therefore \( v_{\mathfrak{P}}(a) \geq 0 \).

Conversely, let \( \mathfrak{P} \in S_p^r(F; a) \) Then \( g_a \in \mathcal{O}_{\mathfrak{P}}[T] \) has a simple zero \( T = 0 \) modulo the maximal ideal of \( \mathcal{O}_{\mathfrak{P}} \), which implies that there exists some \( i \) and \( \mathfrak{Q} \in S_p^r(E_i) \) with \( \mathfrak{P} = \text{res}_{E_i/F}(\mathfrak{Q}) \): Indeed, if \( (F', \nu') \) is a henselization of \( (F, v_{\mathfrak{P}}) \), then \( v' = v_{\mathfrak{P}}' \) for a prime \( \mathfrak{P}' \) of \( F' \), and Hensel’s lemma in the form [9, Thm. 4.1.3(4)] shows that \( g_a \) has a zero in \( F' \), which induces an \( F \)-embedding \( E_i \to F' \), and one can take \( \mathfrak{Q} = \text{res}_{E_i/F}(\mathfrak{P}') \). \( \Box \)

**Theorem 6.4.** For every \( N \in \mathbb{N} \) there exists a 2-dimensional diophantine family \( D_{p,N}^r \) over \( K \) such that

\[
D_{p,N}^r(F) = \{(x, a) \in F^2 \mid v_{\mathfrak{P}}(x) \geq 0 \text{ for every } \mathfrak{P} \in S_p^r(F; a)\}
\]

for every extension \( F/K \) with \( \pi_p^r(F) \leq N \).
Proof. Let $l = 2q^f$. By Theorem 5.9 there exists $N'$ such that for all $E/F/K$ with $[E : F] \leq l$ and $\pi_p^*(F) \leq N$, we have $\pi_p^0(E) \leq N'$, and so

$$R_p^*(E) = R_{p,N'}^*(E).$$  \hfill (6.1)

By Example 3.8, $R_{p,N'}^*$ is a 1-dimensional diophantine family over $K$, and so we may choose polynomials $f_1, \ldots, f_r \in K[X,Y_1,\ldots,Y_m]$ such that

$$R_{p,N'}^*(F) = \{ x \in F \mid \exists y \in F^m : f_1(x,y) = \cdots = f_r(x,y) = 0 \}$$  \hfill (6.2)

for all $F/K$. For each $F/K$ with $\pi_p^*(F) \leq N$, and each $a \in F$, we have

$$F \cap P_{f_1,\ldots,f_r}^*(B_a) = \bigcap_{m \in \text{MaxSpec}(B_a)} \left( F \cap P_{f_1,\ldots,f_r}^*(B_a/m) \right) \quad \text{by Lemma 6.1,}$$

$$= \bigcap_{m \in \text{MaxSpec}(B_a)} \left( F \cap R_p^*(B_a/m) \right) \quad \text{by (6.1) and (6.2),}$$

$$= \bigcap_{\mathfrak{p} \in S_p^*(F,a)} \mathcal{O}_{\mathfrak{p}} \quad \text{by Lemma 6.3,}$$  \hfill (6.3)

where $B_a$ is the $l$-dimensional algebra from Lemma 6.3.

By Lemma 6.2, we may define a 2-dimensional diophantine family $D$ over $K$ satisfying

$$D(F) = \left\{ (x, a) \in F^2 \mid x \in P_{f_1,\ldots,f_r}^*(B_a) \right\}$$

for every extension $F/K$. By (6.3), for every $F/K$ with $\pi_p^*(F) \leq N$ we in fact have

$$D(F) = \left\{ (x, a) \in F^2 \mid x \in \bigcap_{\mathfrak{p} \in S_p^*(F,a)} \mathcal{O}_{\mathfrak{p}} \right\},$$

proving the claim. \hfill $\square$

Corollary 6.5. If $\pi_p^*(F) < \infty$, then for every open-closed set $S \subseteq S_p^*(F)$, the holomorphy ring $\bigcap_{\mathfrak{p} \in S} \mathcal{O}_{\mathfrak{p}}$ is diophantine in $F$.

Proof. As $S$ is open-closed, it is of the form $S_p^*(F,a)$ for some $a \in F$, see [10, Lem. 10.4, 10.5]. Hence the claim follows from Theorem 6.4 and Lemma 3.7. \hfill $\square$

By Example 2.5 this applies in particular to pseudo $p$-adically closed fields like $\mathbb{Q}_p^\dagger$, although for such fields there are in fact simpler ways of establishing Theorem 5.9.

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