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Empirical likelihood estimation of the spatial quantile regression

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Abstract

The spatial quantile regression model is a useful and flexible model for analysis of empirical problems with spatial dimension. This paper introduces an alternative estimator for this model. The properties of the proposed estimator are discussed in a comparative perspective with regard to the other available estimators. Simulation evidence on the small sample properties of the proposed estimator is provided. The proposed estimator is feasible and preferable when the model contains multiple spatial weighting matrices. Furthermore a version of the proposed estimator based on the exponentially tilted empirical likelihood could be beneficial if model misspecification is suspect.

JEL codes: C21, C26

Introduction

Spatial models are quickly gaining prominence in areas such as economics, sociology environmental studies and economic geography building on a long standing tradition in regional studies, real estate and geographical analysis. The popularity of such methods is partially due to the realisation that ‘space’ need not be only geographical, but alternative distance metrics, such as economic and social distances can be employed instead resulting in wide applicability of ‘spatial’ analysis methods. As a result whenever one is concerned with analysis of issues such as social interaction, social norms, social capital, neighbourhood effects, peer group effects, strategic interaction, reference behaviour or yardstick competition theoretical models applying to such concepts are very likely to be susceptible to investigation involving ‘spatial’ methods.

Some examples of theoretical models explicitly considering such issues include the models of increasing returns, path dependence and imperfect competition that underline much of the new economic geography literature (see Fujita et al. 1999 for an overview), neighbourhood spillover effects (Durlauf 1994; Borjas 1995; Glaeser et al. 1996) macroeconomic interaction models (Aoki 1996 and Durlauf 1997) and the interaction framework of Brueckner (2003).

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It is then not surprising that a rich methodological literature dealing with specification, estimation and testing of spatial models have developed. A wide range of statistical methods and models have been developed to allow empirical analysis. This paper contributes to this literature by proposing an alternative estimation method, namely smoothed empirical likelihood, for the recently introduced spatial quantile regression model. The next section briefly reviews the spatial quantile regression. Then we outline the empirical likelihood estimation of the model. The practicalities of the implementation are compared to the alternative estimation methods in terms of advantages and disadvantages in order to help researchers with informal guidelines of which methods would be preferable under what circumstances. Finally we present Monte Carlo study of the finite sample properties of the the proposed estimator.

Spatial quantile regression

Here we will consider the so called spatial quantile regression model. Since this particular term has been used to denote two very different specifications it is important to clarify which one we are referring to. In this paper we follow the definition provided in Kostov (2009). Recently Reich et al. (2011) proposed a totally different model which they also called spatial quantile regression. Their model is a quantile regression with Bernstein polynomial basis representation for the coefficients in which the basis functions are allowed to vary over space. The model of Reich et al. (2011) is a compromise between (Bayesian) non-parametric and parametric approaches to the quantile reression problem. The model of Kostov (2009) on the other hand is fully parametric (frequentist) quantile regression, characterised by endogeneity. The spatial quantile regression model, proposed by Kostov (2009) reads as:

$$y = \lambda(\tau) W_s y + X\beta(\tau) + u \qquad (1)$$

It is a straightforward quantile regression extension of the widely used in spatial econometrics linear spatial lag model. The quantile of the dependent variable *y* is modeled as a function of its spatial lag $W_s y$ (obtained using some predetermined exogenous spatial weighting matrix W_s) and explanatory variables *X*. Hereafter we will treat the spatial weighting matrix as given. It is worth noting that the latter can be viewed as a combination of ‘spatial’ connectedness (neighbourhood structure) and weighting scheme, and defines an interaction pattern amongst the units of analysis. Although discussing the way such interaction pattern is to be determined falls outside the scope of the present paper, the actual metrics system for doing so does not need to be geographical, as the term ‘spatial’ implicitly suggests. Indeed wide range of alternative distance metrics, such as economic, social, technological, cultural etc. can be employed to produce such ‘spatial’ weighting matrices and the implications of the spillovers represented by these can be analysed by this model.

The same model has been studied in Su and Yang (2011) under the name spatial quantile autoregression, a name that makes the link to the mean model that is being generalised, much more explicit. Since in essence the spatial quantile regression is a straightforward generalization of the (mean) spatial lag model, in empirical analysis similarly to the latter, the main point of interest will be the partial derivatives $\left(I-\lambda(\tau)W_s\right)^{-1}\beta(\tau)$, rather than the raw coefficients $\beta(\tau)$. Hereafter, in order to simplify discussion we will ignore this and will focus on estimation issues, concerned with the raw coefficients.

In contrast to the standard (linear) spatial lag model the spatial lag parameter $\lambda(\tau)$ and the vector of regression parameters $\beta(\tau)$ are τ-dependent, where τ is the corresponding quantile of the dependent variable. There are some distinct advantages that this model offers, in particular the ability to approximate (monotonic) nonlinear functions by using an estimator characterised by a parametric rate of convergence (see Kostov, 2009 for more details). Furthermore since the quantile regression method does not involve any assumptions about the error term (in this case *u*), the latter can exhibit arbitrary forms of heteroscedasticity, including spatial dependence, which in the linear spatial modelling is represented by the so called spatial error representation. So the spatial quantile regression can easily combine both spatial lag and spatial error type of dependence in one parsimonious representation. One has to note that although in general (spatial) dependency in the error term does not affect estimators’ consistency (which is also valid for quantile regression estimators), it can have a detrimental effect of efficiency. The latter is however an issue that concerns the construction of confidence intervals. Even in the standard quantile regression with no endogeneity, different methods for construction of confidence intervals can be employed, some of which require stronger assumptions (such as e.g. iid of the errors), while others (e.g. the Monte Carlo Marginal Bootstrap method) do not. (see Koenker 2005 for an overview).

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There is another interesting property of this model. Since this specification allows the spatial parameter $\lambda(\tau)$ to be dependent on τ, it can accommodate a different degree of spatial dependence at different points of the response distribution. For example we could have spatial lag dependence only present in some parts of the distribution of the dependent variable, but not in other. Whether one wants to have a varying degree of spatial dependence or not, depends of course on the particular aims of the analysis and nature of the empirical problem. Therefore this model may not be suitable for every purpose. Note however that it is possible in principle to impose certain restrictions to the model, such as for example making the spatial dependence λ parameter fixed (i.e. common across quantiles) resulting in a restricted version of the model represented by Eq. (1). Here however we will ignore such possibilities and will focus on the original specification for the spatial quantile regression.

Since the spatial lagged variable is present on the right hand side of (1) then, similarly to the mean case, the conventional quantile regression estimator of Koenker and Bassett (1978) will in general be inconsistent. To solve the exogeneity problem Kostov (2009) proposed instrumental quantile regression estimation. In particular two different estimators could be implemented. The first is the two-stage quantile regression estimator of Kim and Muller (2004), implemented in a spatial setting by Zietz et al. (2008). The other is the instrumental variable quantile regression method of Chernozhukov and Hansen (2006), adapted to spatial models in Kostov (2009). While Kostov (2009) implicitly considers only a spatial lag type of dependence, Su and Yang (2011) extend the theoretical arguments of Chernozhukov and Hansen (2006) and prove the applicability of their method under general dependent data design. The two-stage quantile regression estimator of Kim and Muller (2004) also assumes i.i.d. data is not applicable to dependent data.

This paper suggests an alternative estimator, namely a smoothed empirical likelihood estimator for the spatial quantile regression. The next section briefly reviews the empirical likelihood approach and introduces the estimator. It is then compared to the other alternative estimators. Then we present simulation evidence of its performance relative to other spatial estimation methods.

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Empirical likelihood estimator

The empirical likelihood (EL) method (Owen 1988, 1990, 1991) is a nonparametric analogue of likelihood estimation and can be applied to inference of distribution functionals. Its asymptotic variance is the same as that of the efficient GMM (meaning that it is asymptotically efficient). The empirical likelihood (EL) estimator is studied in some detail in Owen (1988), Qin and Lawless (1994) and Imbens (1997). Newey and Smith (2004) showed that the EL estimator and some other related methods, such as the continuous updating estimator (CUE) of Hansen et al. (1996), and the exponential tilting (ET) estimator of Kitamura and Stutzer (1997) and Imbens et al. (1998) all belong to the same family of generalised empirical likelihood (GEL) estimators (although strictly speaking they extend the earlier result of Smith (1997) with regard to the EL and ET). Furthermore Newey and Smith (2004) have proven some interesting theoretical results. These include the fact that in addition to being robust to the number of moment restrictions, the higher order bias of GEL equals the asymptotic bias of the infeasible GMM (i.e. the GMM constructed using the optimal (asymptotic variance minimizing) linear combination).

A major obstacle to the more widespread usage of the above EL type of estimators is the fact that direct optimisation of their objective function can be numerically complicated. This has precluded full-blown Monte Carlo studies of their presumably better finite sample properties. Ramalho (2005) is one of few exceptions with empirical results demonstrating the better performance of the EL type of estimators over the GMM estimators.

To facilitate the discussion consider the following alternative representation of the spatial quantile regression model of Kostov (2009):

$$y_i = d_i'\lambda + X_i'\beta + u_i \quad (2)$$

where for each i= 1,...,n (where n is the sample size) y_i is the dependent variable, X_i is a kx1 vector of exogenous covariates, β is a kx1 vector of (fixed) parameters and similarly d_i and λ are correspondingly a sx1 vectors of endogenous variables and their associated coefficients. In this case the endogeneous variables are simply spatial lags of the dependent variable i.e. $d_i = W y_i$ for some spatial weighting matrix W.

In principle the parameters are varying by quantile and should be indexed by it in Eq. (2) above. Here however for simplicity we will focus our attention on a single (unspecified) quantile and for this reason the usual quantile dependence of the model parameters will be omitted from the notation. We also intentionally use notation indexed by observation in order to facilitate the expression of the associated estimating functions that we will introduce and use to specify the empirical likelihood estimator.

Kostov's (2009) spatial quantile regression model explicitly sets *s*=1, but we will not impose this restriction here. In this way our set up fits the more general quantile regression model with endogenous covariates. Furthermore u_i is an (unobserved) error term that satisfies the usual quantile restriction $P\left[u_i \leq 0 \mid X_i, w_i\right] = \tau$, where τ is the quantile of interest.

Before introducing our approach, consider the quantile regression model with no endogeneity, i.e.

$$y_i = X_i'\beta + u_i \qquad (3)$$

To estimate the model in Eq. (3) we can use the unconditional moment conditions implied by quantile restrictions which are given by:

$$E\left(\left[I\left(y_i \leq X_i'\beta\right) - \tau\right] X_i\right) = 0 \qquad (4)$$

or the conditional (on the exogenous variables) moment conditions:

$$E\left(\left[I\left(y_i \leq X_i'\beta\right) - \tau\right] \mid X_i\right) = 0 \qquad (5)$$

where *I*(.) is the indicator function.

Using any set of moment conditions above GMM type of estimators can be constructed. Note that the most popular linear quantile regression estimator, due to Koenker and Bassett (1978) can be viewed as a GMM type of estimator based on a set of unconditional moment restrictions given in Eq. (4) (see e.g. Otsu, 2003 for details). Empirical likelihood quantile regression estimators using the unconditional moment restrictions, have been studied in Chernozhukov and Hong (2003) and Whang (2006). Otsu (2003, 2008) introduced empirical likelihood estimators based on the conditional restrictions (i.e. on Eq. (5) above), focusing on their first order efficiency.

For a model with endogeneous variables, it is more natural to consider conditional moment restrictions, since conditioning upon the endogeneous variables is necessary to eliminate their influence (in term of the residuals). On the other hand, using instruments to account for endogeneity leads to instrument orthogonality conditions, which have the same form as the unconditional moment conditions.

The estimator of Chernozhukov and Hansen (2006, 2008) can be viewed as an unconditional restrictions GMM type of estimator. They exploit the conventional unconditional moment restriction and the instruments orthogonality restriction:

$$E\left(\left[I\left(y_i - d_i'\lambda \le X_i'\beta\right) - \tau\right] X_i\right) = 0 \qquad (6)$$

$$E\left(\left[I\left(y_i - d_i'\lambda - X_i'\beta \le 0\right) - \tau\right] Z_i\right) = 0 \qquad (7)$$

where Z_i is a vector with some instruments, employed to control for endogeneity. Typically these instruments will be $Z_i = WX_i$, i.e the spatial lags of the independent variables. In the literature GMM estimation often uses higher (normally second order spatial lags. For simplicity here we will restrict ourselves to the first order spatial lags. Since both Kostov (2009) and Su and Yang (2011) use this set of instruments, this would be consistent with previous work in this area. The latter is however by in no way restrictive, since the methods discussed here can use any other alternative set of instruments.

Chernozhukov and Hansen (2006, 2008) consider a model with single endogenous variable and by applying a grid search for the values of the parameter of the endogenous variable (λ which in this case is a scalar) effectively eliminate its influence. Then for each value of λ they apply the quantile regression estimator of Koenker and Bassett (1978), which amounts to using Eq. (6). Finally for the range of λ values they construct they construct a matrix norm that effectively measures the extent to which the orthogonality condition in Eq. (7) is met and select the value of λ which best achieves this. In the notation of Eq. (6) and Eq. (7) above λ is fixed and as a consequence the endogeneity issues disappear. In essence the approach of Chernozhukov and Hansen (2006, 2008) implicitly conditions the otherwise unconditional moment restrictions given in Eq. (6) and Eq. (7) on the endogenous variable d. This conditioning is evaluated over a grid of values for λ. We suggest to implement this conditioning explicitly, by replacing the unconditional moment restrictions by their conditional counterparts.

We can rewrite Eq. (6) and Eq. (7) in the following form:

$$E\left(\left[I\left(y_i \leq d_i'\lambda + X_i'\beta\right) - \tau\right] X_i \mid W_i\right) = 0 \qquad (8)$$

$$E\left(\left[I\left(y_i \leq d_i'\lambda + X_i'\beta\right) - \tau\right] Z_i \mid W_i\right) = 0 \qquad (9)$$

In contrast to the unconditional moment restrictions in Eq. (6) and Eq. (7), here we condition on the endogenous variable(s). This allows us to explicitly consider the endogenous variables in the construction of the moment conditions, rather than 'excluding' them as in Chernozhukov and Hansen (2006, 2008). This amounts to evaluating the conditional moment conditions globally. The above moment conditions as given in Eq. (8) and Eq. (9) can also be viewed as estimating equations to be used to estimate the quantile regression model under endogeneity. The validity of such an approach would in general require identification, which in this particular case can be established with regard to the results of Chernozhukov and Hansen (2006, 2008) since they are using a constrained version of the same moment conditions and therefore the model specified by Eq. (8) and Eq. (9) is properly identified. See Otsu (2003) for detailed condition of this why using the conditional moment conditions identifies the same model specified by the unconditional conditions and increases and is more efficient.

Since the theoretical work concerning EL estimation typically assumes smooth functions of random variables, the indicator function in Eq. (8) and Eq. (9) presents a problem. This can however be overcome by replacing it with a smooth function in such a way that the resulting problem is asymptotically equivalent to the original one (see e.g. Kitamura *et al*., 2004).

Given some bounded, compactly supported (over [-1,1]) and integrated to unity kernel function *K*(.) , by defining $g_h\left(x\right) = \int_{u<x/h} K\left(u\right) du$, a smoothed versions of Eq. (8) and Eq. (9) can be obtained by:

$$E\left(\left[g_h\left(d_i'\lambda + X_i'\beta - y_i\right) - \tau\right] X_i \mid d_i\right) = 0 \qquad (10)$$

and

$$E\left(\left[g_h\left(d_i'\lambda + X_i'\beta - y_i\right) - \tau\right] Z_i \mid d_i\right) = 0 \qquad (11)$$

To simplify notation let us denote by θ the overall parameter vector (i.e. $\theta = \left(\lambda, \beta\right)$) and use $\Upsilon_i\left(\theta\right)$ to denote the functional expression enclosed in the expectations operator in both Eq.

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(10) and Eq. (11) above and omit the conditioning operator. This allows us to write the moment conditions in the following compact form:

$$E\left(\Upsilon_i(\theta)\right)=0 \quad (12)$$

In general the above set of moment restriction could be estimated by:

$$\overline{\Upsilon_i}(\theta)=\frac{1}{n}\sum_{i=1}^{n}\Upsilon_i(\theta)=0 \qquad (13)$$

Using Eq. (13) above to construct an estimator presents a set of equations (which are often linear). Since the number of such moment conditions exceeds the number of observations, there is not a solution to $\overline{\Upsilon_i}(\theta)=0$. However there is a solution to $\tilde{\Upsilon}_i(\theta)=0$, where

$$\tilde{\Upsilon}_i(\theta)=\sum_{i=1}^{n}p_i\Upsilon_i(\theta)=0 \qquad (14)$$

and p_i are the so called implied probabilities associated with the i^{th} observation with $\sum_{i=1}^{n}p_i=1$.

Introducing another *n* (*n* being the sample size) unknowns in the system of equations (14) results in a case where there are more unknowns that equations, resulting in multiple possible solutions. The EL selects among these the one for which the distance between the vector of probabilities *p* and the empirical density 1/*n* is minimised. In the more general Generalised Empirical Likelihood (GEL) type of estimators this distance is represented by a Cressie-Read family of discrepancies. The EL approach, which a particular case of the above mentioned GEL family uses the likelihood ratio to measure this distance.

Then the (smoothed) empirical log likelihood ratio function (also referred to as profile empirical (log) likelihood) would be defined as:

$$\ell_h(\theta)=-2\min_{\sum p_i\Upsilon_i(\theta)=0}\sum_{i=1}^{n}\log\left(np_i\right) \qquad (15)$$

where $p=\left(p_{1,}p_{2,\dots,}p_n\right)'$ is a vector on non-negative weights, summing up to 1.

The Empirical Likelihood estimator can be obtained as a constrained optimisation program, i.e. by minimising the condition given in Eq. (15) subject to

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$\sum_{i=1}^{n} p_i = 1$ and $\sum_{i=1}^{n} p_i \Upsilon_i(\theta) = 0$.

The standard Langrage Multiplier method allows one to derive the optimal weights and replacing them back into the empirical log likelihood ratio expression one can obtain:

$$\ell_h(\theta) = 2\sum_{i=1}^{n} \log\left(1 + t(\theta)' \Upsilon_i(\theta)\right) \qquad (16)$$

where $t(\theta)$ is a vector of Lagrange multipliers.

Then the smoothed empirical likelihood estimator can be defined as:

$$\hat{\theta} = \arg\min_{\theta} \ell_h(\theta) \qquad (17)$$

We also need confidence intervals for the smoothed empirical likelihood (SEL) estimates. Different methods to obtain such confidence intervals are discussed in Whang (2006) and Otsu (2008). These however rely on explicit assumption of just identification, while the spatial quantile regression model we consider here is over-identified. Alternatively we could use a quasi-posterior based on the EL objective function as proposed in Chernozhukov and Hong (2003) and employ Markov chain methods to derive confidence intervals, in a way reminiscent of the Bayesian approach to the problem. Strictly speaking Chernozhukov and Hong (2003) assume i.i.d. data for constructing the appropriate sampler. This assumption can however be relaxed and their suggested approach is applicable to dependent data. Here however we will follow the most generally applicable standard approach to obtaining confidence intervals. Solving Eq. (17) provides us with point estimates for the parameters. Similarly to the parametric likelihood we can obtain confidence intervals by simply contouring the empirical likelihood ratio.

Given the point estimates for each parameter we can apply the following to find the confidence limits at a given confidence level. Here we only describe this for the upper boundary, but the same algorithm is used to obtain the lower boundary.

(i) Take some starting value for the upper boundary.

(ii) Then estimate a restricted quantile regression model in which the value for this parameter is fixed at the current upper boundary.

(iii) Calculate the smoothed empirical likelihood ratio for the restricted model with regard to the full model. Compare with the desired probability level (the ELR follows asymptotically chi–square distribution, see Owen, 1990). Adjust the value for the upper boundary if necessary.

Steps (ii)-(iii) are iterated through until the desired probability level is achieved. This procedure is then repeated for both the upper and lower boundary for each coefficient. In principle step 2 involves repeatedly solving problems like Eq. (17). One can however greatly reduce the computational load and use simpler to calculate estimator, such as the linear programming estimator of Koenker and Bassett (1978) instead in step 2. As for the starting values, these are not crucial, but values that are closer to the reality will reduce the number of search iterations. Therefore one may use the standard errors estimates from a conventional quantile regression routine. Taking into account that the EL intervals can be expected to be more conservative and therefore wider it would be desirable to increase the starting intervals. For example while adding (or subtracting for the lower boundary) two standard errors to the parameter value, could be considered a workable strategy, one may want to increase this to say 3 standard errors (or 2.5 as in the present application) in order to get closer to the final results.

Contouring the empirical likelihood ratio however depends on it being (asymptotically) chi–square distributed. When considering model with dependent data however, although the empirical likelihood estimates remain consistent, the dependency in the data may cause the empirical likelihood ratio to lose it chi-square distribution. (see Kitamura, 1997 for details). Several alternatives to produce confidence intervals could be used. Spatial bootstrap (both parametric and non-parametric) could be employed, but this would in general be undesirable due to the high computational costs. Alternatively the block empirical likelihood approach of Kitamura (1997) can be used.

The blocking idea amounts to treating dependence in a fully nonparametrical way. It is easier to explain it in the time series setting where it was originally introduced. One uses blocks of consecutive observations to provide a nonparametric evaluation of the dependence in the data. The time series blockwise empirical likelihood was first proposed by Kitamura (1997) and Kitamura and Stutzer (1997). It proceeds as follows. First one forms blocks of observations. The aim of forming such blocks is to retain the dependence pattern within each block. At the same time, similarly to the block bootstrap, the dependence structure between the blocks is destroyed. Then one simply applies the 'standard' EL estimation treating the block as if they were i.i.d. observations. The only difference is that the moment conditions i.e. Eq. (10) and Eq. (11) in our representation) are replaced by what Kitamura and Stutzer

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(1997) call the 'smoothed moment function', which is simply the averaged over each block moment function. Kitamura (1997) suggests that more general weighting schemes (in place of simple averaging) could be employed but virtually all related research uses averaging.

The block empirical likelihood estimator has been shown to achieve the same asymptotic efficiency as an optimally weighted GMM for the dependent data case (see Kitamura, 1997 for details). Although the standard and the block empirical likelihood are both consistent and can be used for obtaining point estimates, in the case of weakly dependent data the block EL can be used to provide valid confidence intervals. For weakly dependent quantiles, Chen and Wong (2009) provide theoretical and simulation evidence for the performance of the block empirical likelihood. They also suggest data dependent smoothing to better capture the dependency pattern. Their results are directly relevant in this case since the restriction on the dependency present in the data, namely alpha mixing is the same as the one assumed in Su and Yang (2011).

Note however that here we deal with spatial data. Since the latter is typically characterised by two dimensions, constructing blocks of data that preserve spatial dependence is slightly more involved than in the time series case. Nordman (2008) provides detailed account for block sampling schemes for spatial empirical likelihood for lattice data. These schemes are essentially the same as these used in the spatial block bootstrap and spatial subsampling (see Lahiri, 2003 for a detailed overview of the latter). In essence this requires resampling rectangular blocks of data. The size of these blocks has to be large enough to capture any spatial dependence, but for theoretical reasons should increase more slowly than the sample size. The optimal size of the blocks is an open issue and it often depends on the purpose of the estimation. Probably the most popular method for selecting the block size is the so called 'minimum volatility' method, of Politis et al. (1999, see section 9.3.2). It is a heuristic based on the premise that although some block sizes could be too small or too large, one may expect a whole range of values for the block size to yield an asymptotically correct inference. This is translated into visual inspection of the EL confidence intervals for a range of block sizes to establish a stability region.

The EL estimation proposed here will be valid under the same set of assumptions as in Su and Yang (2011). One important difference is that the method suggested here depends on instrument identification (i.e. their assumption 4(iii)), while for the IVQR this assumption is not necessary, because under weak identification the method is still valid, and efficient inference can be obtained by inverting the Wald type statistic, as suggested in Chernozhukov and Hansen (2008). The cost of dispensing of the above assumption is that the alternative approach will produce wider confidence intervals (see Kostov 2009 for an empirical illustration of this point). For the EL approach outlined here however instrument validity is crucial.

Comparison with alternative estimators

Kim and Muller's (2004) two–step estimator is by any means computationally simpler than the other two approaches. It only requires two consecutive quantile regressions. The method employed in Kostov (2009) and Su and Yang (2011) carries out a search over a set of values for the spatial lag parameter and thus requires a separate quantile regression to be estimated for each value in this range. This puts additional computational load on it. The SEL advocated here requires non-trivial optimisation and hence will in general be computationally expensive.

The three methods have a totally different approach to controlling for endogeneity. Using the terminology of Blundell and Powell (2003), the two stage quantile regression (2SQR) uses the so called 'fitted values' approach, replacing the dependent variable and the endogenous spatially lagged dependent variable by the fitted values from the first stage. The other two methods employed here, on the other hand can be viewed as estimating functions approaches (or 'instrumental variables' approaches in the terminology of Blundell and Powell, 2003). The latter two approaches seek to actively impose the orthogonality conditions for the instruments. In the case of weak identification for example, this would be preferable. In the latter case for the instrumental quantile regression estimator (IVQR) the indirect approach of Chernozhukov and Hansen (2008) based on inverting the Wald test statistic could be employed to ensure valid finite sample inference as demonstrated in Kostov (2009). For the SEL estimator, the general EL higher order asymptotic efficiency arguments apply leading to better finite sample performance, although inference is dependent on the right choice of block size. It is worth mentioning that there is a third approach to endogeneity, namely the 'control function' (CF) approach. For a (non-spatial) quantile regression application of the latter, see Lee (2007).

In the 2SQR and CF approaches, it is required that a reduced-form equation of the endogenous variable (used to produce correspondingly the 'fitted values' and the 'control variable') is valid, and the estimation of the reduced-from equation is consistent. Then the asymptotic distributions of the 2SQR and CF estimators will depend on the method used in the estimation of reduced-form equation of the endogenous variable. In contrast, the instrumental variable estimators (such as the IVQR or the ELQR) are robust to the method of obtaining the estimated instrument. They do not require reduced-form equation of the endogenous variable. In the homoscedastic case all the above estimators could be used to provide consistent estimates. Under heterogeneous effects however the 2SQR estimates could be biased (see Chernozhukov and Hansen, 2006 for the theoretical argument and Kwak, 2010 for simulation evidence). Since in the spatial quantile regression the potential spatial dependence in the data could create such heterogeneity and the conditions under which the 2SQR remains consistent under such circumstances (e.g. constant effect of the endogeneous variable(s) across quantiles) are rather restrictive, we will not pursue the 2SQR in this paper. Of course a quantile regression model with constant spatial lag effect could in some circumstances be a desirable approach, but the model pursued here is much more general.

Another point to consider is the number of endogenous (spatially lagged in this case) variables. The method used in Kostov (2009) is designed to deal with a single spatially lagged variable. For more than one endogenous variable a multidimensional grid search will be required and the latter could be prohibitively expensive from a computational point of view. It can still be feasible if the simulation methods of Chernozhukov and Hong (2003) are used to evaluate the objective function (instead of grid search). Since the latter methods can also be used to evaluate the smoothed empirical likelihood (see the confidence intervals discussion in previous section) this provides an interesting estimation alternative applicable for both estimators.

The 2SQR, the CF approach and the SEL on the other hand can deal with an arbitrary number of endogenous variables. In the spatial context, models with several spatial weighting matrices can arise naturally, see e.g. LeSage and Fischer (2012) where different 'spatial' weighting matrices account for static and dynamic spillovers. Larger number of endogeneous variables will increase the number of moment conditions to be included in the SEL objective function hence increasing the computational costs. The largest computational cost to the SEL estimator however comes from the sample size. Since one needs to evaluate multiple smoothed quantile score functions, such an evaluation could be computationally demanding when large samples are employed. Note however, that the explicit motivation of the Kostov's (2009) original proposal is obtaining a flexible model with small sample sizes. In the light of this non-parametric modelling alternative may be preferable for problems with larger sample sizes.

The SEL proposed here could be further improved by employing Barlett type corrections and/or bootstrap. Furthermore some more general GEL or pseudo-empirical likelihood methods could be implemented to the same problem. In particular the robust to misspecification Exponentially Tilted Empirical Likelihood (ETEL) method of Schennach (2007) could be of potential interest. The later was originally proposed from a Bayesian point of view (Schennach, 2005) and an adaptation for quantile regression has been implemented in Lancaster and Jun (2010) showing promising potential. An interesting potential advantage of EL methods with many spatially lagged variables is that EL ratio tests could be designed or the general over-identification tests could be adapted to test the significance of a subset of spatially lagged variables, providing means for endogenous variables selection. The latter would of course only be feasible when the number of such variables is relatively small, because of the potentially considerable computational costs.

Simulation study

In order to investigate the finite sample performance of the proposed estimator, we carried out a small simulation study. In this study we consider the direct estimates of Eq. (1). Note that in real word problems analysis correct interpretation of the model parameters would require computation of the partial derivatives, as suggested in LeSage and Fischer (2008). For the spatial quantile regression such a computation can be done similarly to the linear spatial lag model. In order to simplify matters here we will treat these partial derivatives as unknown.

In principle we follow the simulation design of Su and Yang (2011) with some modifications. In addition to the empirical likelihood estimator we considered the instrumental variable quantile regression estimator of Chernozhukov and Hansen (2006, 2008) as well as the exponentially tilted empirical likelihood (ETEL) estimator of Schennach (2007), adapted to quantile regression in the same way as the proposed estimator. Since the ETEL estimator is in principle robust to misspecification, it would be interesting to look at its finite sample performance for the spatial model. Although in the simulation design we do not consider misspecification, it is nevertheless useful to consider the cost that such robust estimators pay whenever misspecification is absent. Additionally we also consider two mean estimator, namely the spatial two-stage least squares (STSLS) of Kelejian and Prucha (1998) and the (quasi) Maximum Likelihood (QML) estimator.

The STSLS is probably the simplest, least computationally expensive estimator for spatial lag type of models, while the QML is still the most widely applicable such estimator.

The specific DGP used in the simulations follows Su and Yang (2011):

$$y_i = \lambda(\nu_i) w_i + \beta(\nu_i)' x_i \qquad (18)$$

where w_i is the spatially lagged dependent variable, $x_i = (1, x_i^0)$ and $x_i^0 \sim iid\ N(0,1)$.

$$\lambda(\nu_i) = 0.5 + 0.1 F^{-1}(\nu_i)$$

$$\beta(\nu_i) = (2.0, 1.0)' + (0.5, 0.5)' F^{-1}(\nu_i)$$

with $\nu_i \sim iid\, U(0,1)$ and $F(.)$ is a probability distribution function. In the first set of simulations we use the standard normal distribution, while in the next one we use a Student-t distribution with 3 degrees of freedom. We denote the former as DGP1 and the latter as DGP2. The above setup simulates a quantile regression process in which the linear quantile restriction is met (see Su and Yang, 2011 for details) and we will not discuss it here. We simulate the spatial weighting matrix under rook contiguity again following Su and Yang (2011). This is implemented as follows:

(i). Generate randomly *n* integers (*n* is the sample size) from 1 to *n* without replacement.

(ii). Arrange the above numbers in a matrix with ten rows (Su and Yang, 2011 use only five rows).

(iii). Regarding the matrix created in step 2 above as a spatial locations grid and the numbers populating it as the corresponding observation numbers create a neighbourhood matrix according to rook contiguity

(iv). Row standardise the spatial weighting matrix created in step 3 above.

We simulate samples with size of 100, 200 and 500 observations. A different spatial weighting matrix is simulated within each iteration. Each set of results is based on evaluating 1,000 Monte Carlo replications. Note that in the above simulation design the median and the mean for each of the parameters coincide. This allows one to compare the median quantile regression results with the two mean regression estimators employed in this study.

Tables 1-3 present the empirical (i.e. Monte Carlo) bias together with the standard deviation and the root mean squared errors (RMSE) for each of the compared estimators at the 0.5th , 0.25th and the 0.75th quantile. Where an estimator is applied to a quantile different from the median, the acual quantile is added to its description (i.e. ELQR0.25 refers to the EL estimator applied to the 0.25th quantile). For reporting purposes we standardise the empirical bias by dividing it by the true value of the corresponding parameters. This allows for direct comparison of the results across different parameters. The standard deviation and the RMSE are calculated from the standardised bias and as such are also directly comparable.

Simulation Results

Following similar simulation design to that of Su and Yang (2011) allows for comparison with their results. The results presented here seem to show slightly higher empirical bias, as well as standard deviation and RMSE. This is due to on the one hand the slightly different design for the rook contiguity matrix with ten rather than five rows, which results in a slightly ‘denser’ interaction pattern. The other reason (the random number generation notwithstanding) for such differences could be the fact that we simulate different spatial weighting matrix in each iteration. (Su and Yang (2011) are unclear about whether they use the same spatial weighting matrix in their replications). In general the RMSE decreases across all estimators, as the sample size increases, which is to be expected.

The first point of interest is the comparative performance of the EL type of estimators for the 0.5th quantile with regard to the conventional mean regression estimators. Although, the median-based quantile estimators generally perform better than the mean regression estimator, interestingly they are more dispersed (higher standard deviation and RMSE) than the QML for the small sample size (n=100). The additional variability introduced by simulating different spatial weighting matrices could have contributed to such a slower convergence when the sample size is smaller. The overriding conclusion from this comparison is that the quantile estimation methods are comparable to the mean regression method for spatial lag models.

The other, more important issue, to consider is the relative performance of the suggested estimator against the IVQR at different quantiles. In general the EL estimator performs very similarly to the IVQR in terms of both bias and RMSE. The ETEL estimators seems to be slightly worse than the EL one in terms of empirical bias, but tends to perform slightly better in terms of RMSE under non-normal heteroscedastic errors (i.e. DGP2). This suggests that in some cases the additional computational costs associated with EL estimators may not bring in sufficient advantages to justify their usage over the IVQR estimator. Yet, as already discussed, whenever the IVQR is difficult to implement, as e.g. in models with multiple spatial weighting matrices, it is preferable, and given that it has similar properties to the IVQR estimator, it can be a more effective alternative.

Another interesting conclusion from the simulation results, that may require further investigation is the fact that although being outperformed by the other quantile estimators, the ETEL estimator is still performing reasonably well. Given that it is robust with regard to misspecification, it looks like that it pays a relatively small price for such robustness. It could therefore be interesting to consider the issue of misspecification and the comparable performance of the alternative estimators in such circumstances. We can nevertheless tentatively suggest the when a misspecification is suspect, it might be warranted to undertake ETEL estimation, i.e. to consider the additional computational costs as a trade-off for limiting its implications on the results. The exact consequences of such a choice will need to be investigated separately and is beyond the scope of the present study.

Conclusions

This paper suggested an alternative smoothed empirical likelihood estimator for the spatial quantile regression model. Although this estimator can be computationally demanding for larger sample sizes, it is feasible and practical for small and medium size samples typical for spatial dependence problems, where the parametric rate of convergence of the quantile regression estimators can be most beneficial. The main computational cost of the proposed procedure lies in contouring the empirical likelihood in order to obtain confidence intervals. Therefore alternative methods for achieving this task, outlined in the methodology section, could reduce its computational costs and increase its appeal.

One of the main advantages of our proposal however lies in establishing a general estimation framework in which the empirical likelihood can be substituted by alternative estimation criteria (most notably those of the generalised empirical likelihood family) resulting in a range of estimators that should be able to achieve higher order improvements compared to more standard (generalised) method of moments approaches.

An added benefit of the proposed method is the ability to handle multiple spatial weighting matrices, which is difficult for the IVQR methods. Furthermore the simulation evidence suggests that the ETEL version of the estimator, pays relatively small price (in terms of efficiency) for allowing for potential misspecification. Although the latter requires further investigation, it tentatively suggests that this could be a preferred estimation option in empirical investigations when there are doubts about the correct model specification.

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Table 1. Simulation Results (sample size 100)

	DGP1			DGP2		
	λ	β_1	β_2	λ	β_1	β_2
			Bias			
stsls	-0.0314	0.0356	-0.0026	-0.0379	0.0585	0.0068
QML	-0.0362	0.0410	0.0054	-0.0355	0.0517	0.0198
ivqr	-0.0153	0.0135	-0.0078	-0.0237	0.0198	-0.0107
ELQR	-0.0125	0.0113	-0.0076	-0.0285	0.0227	-0.0111
ETELQR	-0.0264	0.0278	-0.0012	-0.0374	0.0367	-0.0108
ivqr0.25	-0.0012	-0.0027	0.0834	-0.0318	0.0730	0.0637
ELQR0.25	0.0036	-0.0104	0.0911	-0.0327	0.0704	0.0685
ETELQR0.25	-0.0084	0.0028	0.0955	-0.0381	0.0929	0.0719
ivqr0.75	-0.0330	0.0345	-0.0503	-0.0198	0.0443	-0.0685
ELQR0.75	-0.0343	0.0372	-0.0543	-0.0290	0.0537	-0.0694
ETELQR0.75	-0.0415	0.0447	-0.0538	-0.0355	0.0581	-0.0688
			SD			
stsls	0.1039	0.1035	0.0794	0.1052	0.1092	0.0995
QML	0.0555	0.0556	0.0775	0.0391	0.0434	0.0994
ivqr	0.0910	0.0911	0.0617	0.0859	0.0970	0.0457
ELQR	0.0905	0.0900	0.0609	0.1020	0.0969	0.0457
ETELQR	0.0927	0.0922	0.0633	0.0878	0.0868	0.0442
ivqr0.25	0.1087	0.0918	0.0656	0.1043	0.0898	0.0554
ELQR0.25	0.1062	0.0890	0.0636	0.1100	0.0904	0.0547
ETELQR0.25	0.1086	0.0915	0.0644	0.1125	0.0919	0.0555
ivqr0.75	0.1164	0.0869	0.0742	0.1098	0.0801	0.0575
ELQR0.75	0.1145	0.0848	0.0723	0.1071	0.0792	0.0563
ETELQR0.75	0.1162	0.0862	0.0727	0.1084	0.0798	0.0568
			RMSE			
stsls	0.1043	0.1040	0.0793	0.1054	0.1096	0.0995
QML	0.0565	0.0570	0.0775	0.0404	0.0459	0.1001
ivqr	0.0911	0.0912	0.0619	0.0860	0.0971	0.0458
ELQR	0.0905	0.0901	0.0611	0.1020	0.0969	0.0459
ETELQR	0.0930	0.0925	0.0633	0.0879	0.0869	0.0443
ivqr0.25	0.1086	0.0917	0.0735	0.1049	0.0899	0.0600
ELQR0.25	0.1061	0.0890	0.0732	0.1106	0.0905	0.0600
ETELQR0.25	0.1086	0.0914	0.0747	0.1132	0.0922	0.0613
ivqr0.75	0.1169	0.0873	0.0845	0.1099	0.0803	0.0669
ELQR0.75	0.1151	0.0853	0.0845	0.1072	0.0796	0.0661
ETELQR0.75	0.1170	0.0869	0.0846	0.1086	0.0802	0.0664

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Table 2. Simulation Results (sample size 200)

	DGP1			DGP2		
	λ	β_1	β_2	λ	β_1	β_2
			Bias			
stsls	-0.0213	0.0285	0.0047	-0.0492	0.0686	0.0124
QML	-0.0230	0.0301	0.0084	-0.0196	0.0380	0.0226
ivqr	-0.0073	0.0084	-0.0007	-0.0177	0.0189	-0.0034
ELQR	-0.0101	0.0116	-0.0001	-0.0212	0.0210	-0.0030
ETELQR	-0.0172	0.0209	0.0039	-0.0303	0.0353	0.0021
ivqr0.25	0.0138	-0.0139	0.0757	-0.0196	0.0062	0.0615
ELQR0.25	0.0134	-0.0174	0.0866	-0.0238	0.0065	0.0676
ETELQR0.25	0.0050	-0.0076	0.0886	-0.0314	0.0158	0.0701
ivqr0.75	-0.0238	0.0248	-0.0394	-0.0189	0.0301	-0.0537
ELQR0.75	-0.0284	0.0323	-0.0441	-0.0275	0.0384	-0.0569
ETELQR0.75	-0.0313	0.0345	-0.0443	-0.0324	0.0446	-0.0563
			SD			
stsls	0.0720	0.0722	0.0571	0.1327	0.1351	0.1189
QML	0.0413	0.0416	0.0558	0.0464	0.0510	0.1113
ivqr	0.0578	0.0574	0.0412	0.0782	0.0776	0.0458
ELQR	0.0577	0.0573	0.0418	0.0783	0.0770	0.0461
ETELQR	0.0587	0.0580	0.0429	0.0800	0.0796	0.0505
ivqr0.25	0.0708	0.0593	0.0464	0.1066	0.0908	0.0577
ELQR0.25	0.0684	0.0574	0.0454	0.1073	0.0905	0.0565
ETELQR0.25	0.0705	0.0593	0.0462	0.1086	0.0917	0.0568
ivqr0.75	0.0691	0.0510	0.0478	0.1026	0.0727	0.0641
ELQR0.75	0.0670	0.0495	0.0462	0.1000	0.0720	0.0624
ETELQR0.75	0.0681	0.0504	0.0459	0.1019	0.0731	0.0629
			RMSE			
stsls	0.0723	0.0726	0.0571	0.1335	0.1367	0.1191
QML	0.0419	0.0425	0.0560	0.0468	0.0522	0.1121
ivqr	0.0578	0.0574	0.0412	0.0783	0.0778	0.0459
ELQR	0.0578	0.0574	0.0417	0.0785	0.0772	0.0462
ETELQR	0.0589	0.0583	0.0430	0.0805	0.0803	0.0505
ivqr0.25	0.0709	0.0594	0.0553	0.1067	0.0908	0.0686
ELQR0.25	0.0685	0.0575	0.0570	0.1074	0.0905	0.0696
ETELQR0.25	0.0705	0.0593	0.0582	0.1089	0.0917	0.0708
ivqr0.75	0.0695	0.0514	0.0573	0.1028	0.0730	0.0781
ELQR0.75	0.0677	0.0501	0.0582	0.1005	0.0726	0.0783
ETELQR0.75	0.0689	0.0511	0.0581	0.1025	0.0739	0.0784

Table 3. Simulation Results (sample size 500)

	DGP1			DGP2		
	λ	β_1	β_2	λ	β_1	β_2
				Bias		
stsls	-0.0080	0.0154	0.0068	-0.0126	0.0317	0.0185
QML	-0.0040	0.0113	0.0081	0.0002	0.0207	0.0249
ivqr	-0.0047	0.0056	0.0011	-0.0025	0.0016	-0.0027
ELQR	-0.0067	0.0079	0.0016	-0.0058	0.0055	-0.0024
ETELQR	-0.0144	0.0162	0.0033	-0.0140	0.0152	0.0011
ivqr0.25	0.0100	-0.0083	0.0715	0.0066	-0.0278	0.0582
ELQR0.25	0.0074	-0.0100	0.0871	0.0036	-0.0254	0.0671
ETELQR0.25	0.0033	-0.0042	0.0885	0.0023	-0.0175	0.0684
ivqr0.75	-0.0185	0.0172	-0.0354	-0.0130	0.0141	-0.0464
ELQR0.75	-0.0232	0.0256	-0.0418	-0.0175	0.0222	-0.0510
ETELQR0.75	-0.0258	0.0279	-0.0417	-0.0228	0.0275	-0.0503
				SD		
stsls	0.0751	0.3011	0.0575	0.0822	0.0832	0.0714
QML	0.0450	0.1792	0.0570	0.0324	0.0352	0.0880
ivqr	0.0520	0.2070	0.0403	0.0375	0.0370	0.0279
ELQR	0.0532	0.2119	0.0417	0.0387	0.0383	0.0290
ETELQR	0.0539	0.2124	0.0424	0.0387	0.0380	0.0296
ivqr0.25	0.0646	0.0904	0.0457	0.0536	0.0469	0.0344
ELQR0.25	0.0631	0.0883	0.0440	0.0529	0.0461	0.0336
ETELQR0.25	0.0639	0.0896	0.0444	0.0537	0.0469	0.0341
ivqr0.75	0.0649	0.0478	0.0480	0.0535	0.0381	0.0389
ELQR0.75	0.0635	0.0467	0.0470	0.0532	0.0379	0.0378
ETELQR0.75	0.0644	0.0476	0.0470	0.0536	0.0382	0.0378
				RMSE		
stsls	0.0752	0.3025	0.0579	0.0823	0.0837	0.0722
QML	0.0450	0.1805	0.0576	0.0324	0.0357	0.0892
ivqr	0.0520	0.2072	0.0403	0.0375	0.0369	0.0279
ELQR	0.0533	0.2124	0.0417	0.0387	0.0383	0.0290
ETELQR	0.0544	0.2147	0.0425	0.0389	0.0382	0.0296
ivqr0.25	0.0648	0.0905	0.0658	0.0537	0.0470	0.0490
ELQR0.25	0.0632	0.0885	0.0726	0.0529	0.0462	0.0525
ETELQR0.25	0.0638	0.0896	0.0735	0.0537	0.0470	0.0534
ivqr0.75	0.0657	0.0483	0.0674	0.0537	0.0383	0.0548
ELQR0.75	0.0648	0.0480	0.0730	0.0536	0.0383	0.0568
ETELQR0.75	0.0660	0.0490	0.0729	0.0541	0.0388	0.0563