Measurability in Modules

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Abstract

In this paper we prove that in modules, MS-measurability (in the sense of Macpherson-Steinhorn) depends on being able to define a measure function on the p.p. definable subgroups. We give a classification of abelian groups in terms of measurability. Finally we discuss the relation with $\mathbb{Q}[t]$-valued measures.

1 Introduction

A structure M is MS-measurable (in the sense of Macpherson-Steinhorn) if one can definably assign an ($\mathbb{N}$-valued) dimension and an ($\mathbb{R}$-valued) measure to each definable set in M. This assignment must obey some basic axioms [Definition 2.2]. The expression measurability in a mathematical context carries with it a lot of baggage and nuance, so we prefer to refer to this notion of Macpherson and Steinhorn as *MS-measurability.*

The motivating examples of MS-measurable structures are pseudofinite fields (or ultraproducts of finite fields). In [6] Macpherson and Steinhorn generalise from the specific case of finite fields, developing a notion of dimension and measure for definable subsets of finite structures. A one-dimensional asymptotic class is a collection of finite L-structures [Definition A.3], for some language L, to which one can assign (in a definable way) a dimension d and measure μ such that for any formula $\phi(\bar{x},\bar{y}) \in L$ and tuple $\bar{a}$ of M of suitable length, $||\phi(M^n,\bar{a})| - \mu|M|^d| \leq C|M|^{d-\frac{1}{2}}$, where M is any finite structure in the collection, and C a positive constant.

In [3] Elwes and Macpherson develop the more general notion of an N-dimensional asymptotic class. They prove any ultraproduct of an N-dimensional asymptotic class is MS-measurable. As well as finite fields, finite cyclic groups are also an example of an asymptotic class, and therefore ultraproducts of finite cyclic groups are MS-measurable (this is used in Section 4). However, it is not the case that all MS-measurable structures are ultraproducts of asymptotic classes. We know, for example, that vector spaces are MS-measurable in this sense. Any MS-measurable structure is supersimple of finite SU-rank, in fact the dimension behaves essentially like SU-rank.

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In this paper we consider MS-measurability in the case of modules, which are stable, and therefore MS-measurable modules are superstable. It is well known [7] that in modules every formula is equivalent to a Boolean combination of positive primitive (p.p.)-formulas [Definition 2.6]. In Section 2 we recall some of the main facts of the model theory of modules, as well as introducing MS-measurability in detail. In Section 3 we use these facts to show our main result [Theorem 3.1] that MS-measurability of a complete theory of modules relies entirely on properties of the subgroups defined by p.p. formulas without parameters.

In Section 4 we restrict our attention to abelian groups (or $\mathbb{Z}$-modules) and classify the MS-measurable abelian groups. We also remark that the MS-measurable abelian groups are precisely the pseudofinite abelian groups, where a pseudofinite structure is one which is infinite and elementarily equivalent to an ultraproduct of finite structures. Section 5 makes connections with the notion of $\mathbb{Q}[t]$-valued measures on Boolean combinations of cosets of $\mathbb{Z}^n$ defined in [1].

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2 Preliminaries

Throughout, unless otherwise specified, T is a complete theory in signature L, M a model of T.

2.1 MS-measurability

Definition 2.1. *Def(M) is the set of definable (with parameters) sets in M.*

In [3] Elwes and Macpherson give the following definition (which is equivalent to the original definition in [6], but drops the assumption of finite D-rank).

Definition 2.2. *An infinite L-structure M is* MS-MEASURABLE *if there is a function $h : Def(M) \longmapsto \mathbb{N}\times\mathbb{R}^{>0}\cup\{(0,0)\}$ (we write $h(X) = (dim(X), meas(X))$ such that h satisfies the following:*

1. *For each L-formula $\varphi(\bar{x}, \bar{y})$ there is a finite set $D_\varphi \subset \mathbb{N} \times \mathbb{R}^{>0} \cup \{(0,0)\}$ such that for all $\bar{a} \in M^m$ we have $h(\varphi(M^n, \bar{a})) \in D_\varphi$.*

2. *If $\varphi(M^n, \bar{a})$ is finite then $h(\varphi(M^n, \bar{a})) = (0, |\varphi(M^n, \bar{a})|)$.*

3. *For every L-formula $\varphi(\bar{x}, \bar{y})$ and all $(d, \mu) \in D_\varphi$, the set $\{\bar{a} \in M^m : h(\varphi(M^n, \bar{a})) = (d, \mu)\}$ is $\emptyset$-definable.*

4. *(Fubini) Let $X, Y \in Def(M)$ and $f : X \mapsto Y$ be a definable surjection. Then there is an $r \in \omega$ and $(d_1, \mu_1), ..., (d_r, \mu_r) \in \mathbb{N} \times \mathbb{R}^{>0} \cup \{(0,0)\}$ such that if $Y_i = \{\bar{y} \in Y : h(f^{-1}(\bar{y})) = (d_i, \mu_i)\}$ then $Y = Y_1 \cup ... \cup Y_r$ is a partition of Y into non-empty disjoint definable sets.*

   Moreover, let $h(Y_i) = (e_i, \nu_i)$ for $i \in \{1, ..., r\}$ and $c := Max\{d_1 + e_1, ..., d_r + e_r\}$. Suppose that this maximum is attained by $d_1 + e_1, ..., d_s + e_s$ for some $s \le r$. Then $h(X) = (c, \mu_1 \cdot \nu_1 + ... + \mu_s \cdot \nu_s)$.

If $X \in Def(M)$ and $h(X) = (d, \mu)$, we call d the MS-DIMENSION of X and μ the MS-MEASURE of X, and h the MS-MEASURING FUNCTION.

Remark 2.3. *1. If M is MS-measurable, then every $N \models Th(M)$ will be MS-measurable (by clause (iii), see 3.7 in [3]). So we call a complete theory T MS-measurable if it has an MS-measurable model.*
2. Any sets in definable bijection will clearly have the same MS-dimension and MS-measure (clause(iv)).

Remark 2.4. *Suppose X is a definable set in an MS-measurable structure. Then $dim(X) = 0$ if and only if X is finite.*

In [6] (Theorem 5.7) the following is proved. Note that in the first clause x is a single variable. This reduces the number of cases we need to consider in Theorem 3.1.

Theorem 2.5. *Let $h : Def(M) \longmapsto \mathbb{N} \times \mathbb{R}^{>0} \cup \{(0,0)\}$ with $h(X) = (dim(X), meas(X))$ satisfy the following:*

(i) For each L-formula $\phi(x, \bar{y})$ there is a finite set $D_\phi \subset \mathbb{N} \times \mathbb{R}^{>0}$, such that for all $\bar{a} \in M^m$ if $\phi(x, \bar{a}) \neq \emptyset$ we have $h(\phi(M, \bar{a})) \in D_\phi$. For each $(n_i, \mu_i) \in D_\phi$ the set $\{\bar{y} : h(\phi(M, \bar{y})) = (n_i, \mu_i)\}$ is $\emptyset$-definable in M.

(ii) For each $n \in \omega$ and $\bar{a} \in M^n$ we have $h(\{\bar{a}\}) = (0, 1)$.

(iii) For all $n \in \omega$, and all disjoint definable sets $X_1, X_2 \subseteq M^n$ we have that:

$$meas(X_1 \cup X_2) = \begin{cases} meas(X_1) + meas(X_2) & \text{if } dim(X_1) = dim(X_2) \\ meas(X_1) & \text{if } dim(X_1) > dim(X_2) \\ meas(X_2) & \text{if } dim(X_1) < dim(X_2) \end{cases}$$

(iv) For each $n \in \omega$ and $i \in \{1...n\}$ the following holds:
Let $X \subset M^n$ be definable, $\pi : M^n \to M$ be the projection onto the i^{th} co-ordinate. Suppose there is a (d, μ) such that $\forall a \in \pi(X)$ we have $h(\pi^{-1}(a) \cap X) = (d, \mu)$. Then $dim(X) = dim(\pi(X)) + d$ and $meas(X) = meas(\pi(X)) \times \mu$.

Then M is MS-measurable.

2.2 Model theory of modules

Throughout, unless otherwise specified, M is a left R-module over a ring R and $L = L_R = \{+, 0, r\}_{r \in R}$ is the language of left R-modules.

Definition 2.6. POSITIVE PRIMITIVE (P.P.) FORMULAS *(without parameters) are formulas of the form $\exists \bar{y}(\psi_1(\bar{x}, \bar{y}) \wedge ... \wedge \psi_k(\bar{x}, \bar{y}))$ where $\psi_i(\bar{x}, \bar{y})$ are atomic formulas. In the language of modules these are of the form $\exists w_1....w_k \bigwedge_{j=1}^{m} (\Sigma r_{ij} v_i + \Sigma s_{lj} w_l = \bar{0})$ where $r_{ij}, s_{lj} \in R$.*

Remark 2.7. *The set of p.p. formulas is closed under conjunction (up to logical equivalence).*

Remark 2.8. *1. Let $\psi(\bar{x})$ be a p.p. formula (without parameters) in the language of modules. Then $\psi(\bar{x})$ will define a subgroup of M^n (where n is the length of $\bar{x}$). We call the subgroups these define* P.P. SUBGROUPS *in* M.

2. *Let $\psi(\bar{x}, \bar{y})$ be a p.p. formula (without parameters) in the language of modules. If $\bar{a} \in M^m$, then $\psi(\bar{x}, \bar{a})$ if non-empty will define a coset of $\psi(\bar{x}, \bar{0})$.*

3. *Let $f : X \longrightarrow Y$ be a surjective function such that f (i.e. its graph), X and Y are all p.p. definable without parameters. Let $\bar{y} \in Y$. Then*

   (a) *$f^{-1}(\bar{y})$ is a coset of $f^{-1}(\bar{0}) = ker(f)$ (by 2.). So f is in fact a homomorphism.*

   (b) *Suppose M is MS-measurable. Then Remark 2.3(2) gives that all cosets must have the same MS-dimension and MS-measure. That is to say, for $\bar{y} \in Y$, $h(f^{-1}(\bar{y})) = h(f^{-1}(\bar{0}))$ and $Y = \{\bar{y} \in Y : h(f^{-1}(\bar{y})) = h(ker(f))\}$.*

Definition 2.9. *Let $\phi_1(\bar{x})$ and $\phi_2(\bar{x})$ be p.p. formulas defining p.p. subgroups G and H respectively. The value of $|G : H \cap G|$ will depend on which complete theory of modules we are working in. An* INVARIANT SENTENCE *is one which expresses a fact of the form $|G : H \cap G| \leq m$, where H and G are p.p. subgroups of M^n and m a positive integer. These are first order sentences, and therefore if we work in a complete theory we fix which invariant sentences are true.*

Theorem 2.10. *(Baur, Monk, also see [7], [5]) In the language of left R-modules for every formula $\phi(\bar{x})$(without parameters) of L there is a formula $\psi(\bar{x})$ which is a boolean combination of p.p. formulas and invariant sentences such that $\models \phi(\bar{x}) \leftrightarrow \psi(\bar{x})$.*

Note: If T is a complete theory then the invariant sentences are fixed, so every formula is equivalent to Boolean combination of p.p. formulas. If we allow parameters in the formulas we get that every definable set is equivalent to a boolean combination of cosets of p.p. definable subgroups.

Remark 2.11. *All definable (with parameters) sets in M are defined by formulas of the form:*

$$\bigvee_{i=1}^{n} (\phi_{i0}(\bar{x}, \bar{a}_{i0}) \wedge (\bigwedge_{j=1}^{n_i} \neg\phi_{ij}(\bar{x}, \bar{a}_{ij})))$$

where $\phi_{ij}(\bar{x}, \bar{y})$ are p.p., and $M \models \phi_{ij}(\bar{x}, \bar{a}_{ij}) \rightarrow \phi_{i0}(\bar{x}, \bar{a}_{i0})$ for all j.

3 Results and Proofs

Let $Def_{p.p.}(M)$ be the set of subgroups defined by p.p. formulas (without parameters).

Theorem 3.1. *Let M be a module. Suppose we have a function*

$$h_p : Def_{p.p.}(M) \to \mathbb{N} \times \mathbb{R}^{>0} \cup \{(0,0)\}$$

such that the following hold:

(a) If $|X|$ is finite, then $h_p(X) = (0, |X|)$.

(b) For $X, Y \in Def_{p.p.}(M)$, and $f : X \to Y$ a surjective $\emptyset$-p.p. definable homomorphism, if $h_p(Y) = (d, \mu)$ and $h_p(ker(f)) = (e, \nu)$ then $h_p(X) = (d + e, \mu\nu)$.

(c) Let X and Y be p.p. subgroups with $X \supseteq Y$, with $h_p(Y) = (d, \nu)$. Then

1. If $|X : Y| = n < \omega$ then $h_p(X) = (d, n \cdot \nu)$.

2. If $|X : Y|$ is infinite then $h_p(X) = (d', \nu')$ for some $d' > m$.

Then h_p extends uniquely to an MS-measuring function on the whole of Def (M).

Proof: The proof of this theorem is rather long, so we begin by summarizing the main steps, highlighting the main difficulties.

(1) *Step one* is to assign an MS-dimension and MS-measure to all definable sets. We do this by using the description of definable sets given by Remark 2.11. The difficulty here is to give a coherent MS-dimension and MS-measure to sets defined using disjunctions. We introduce the idea of islands of co-measurability, and their building blocks, to deal with this. The aim is then go on to show that this assignment obeys clauses (i)-(iv) of Theorem 2.5, thereby showing that the assignment does in fact give an MS-measuring function.

(2) *Step two* is to remark that (ii) and (iii) of Theorem 2.5 are obvious from the way we assign MS-dimension and MS-measure.

(3) *Step three* establishes clause (i), i.e. we show firstly that for each formula $\phi(x, \bar{y})$ the set $\{h(\phi(x, \bar{a})) : \bar{a} \in M^n\}$ is finite, and secondly this assignment is $\emptyset$-definable. We do this by reducing to formulas containing a single island of co-measurability, and use a result from [4].

(4) Finally, in *step four*, we show clause (iv) of Theorem 2.5. The way this is stated allows us to reduce to the case where the function is a projection with fibres of constant MS-measure. We further reduce to considering projections of definable sets containing a single island of co-measurability, the result then follows from a simple counting argument on indices of building blocks. The steps in the proof refer to the ones given in this paragraph.

Before we begin the proof we need to fix some conventions for definable sets. For an L-formula $\psi(\bar{x}, \bar{y})$ and tuple $\bar{a}$ from M of appropriate length we use the following notation:

$$\begin{aligned}
\psi(\bar{x},\bar{y}) &= \bigvee_{i=1}^{n}(\phi_{i0}(\bar{x},\bar{y}) \wedge \bigwedge_{j=1}^{n_i} \neg\phi_{ij}(\bar{x},\bar{y})),\\
K_{ij} &= \{\bar{x} \in M : M \models \phi_{ij}(\bar{x},\bar{0})\},\\
X_{ij} &= \{\bar{x} \in M : M \models \phi_{ij}(\bar{x},\bar{a})\}\\
X_i &= X_{i0} \setminus (X_{i1} \cup ... \cup X_{in_i})\\
X &= \bigcup_{i=1}^{n} X_i = \bigcup_{i=1}^{n}(X_{i0} \setminus (X_{i1} \cup ... \cup X_{in_i}))
\end{aligned}$$

Note: The X_{ij}'s are dependent on $\bar{a}$, whereas the K_{ij} are not. By Remark 2.11 every formula will be equivalent to one of the appropriate form, so we can use this notation for every definable set X. Also, by the same Remark, $X_{ij} \subseteq X_{i0}$.

In the following we will let $h_p = (dim_p, meas_p)$. For a p.p. subgroup X, $dim_p(X)$, $meas_p(X)$ will be referred to as p-dimension and p-measure respectively.

Islands of co-measurability

It will be convenient for our analysis to separate definable sets into "islands of co-measurability".

Definition 3.2. *Suppose we have the definable set $X = \bigcup_{i=0}^{n}(X_{i0} \setminus \cup_{j=1}^{n_i} X_{ij}) = \bigcup_{i=0}^{n} X_i$, and suppose that $dim_p(K_{i0}) = dim_p(K_{\iota 0})$ for all $0 \leq i, \iota \leq n$. Consider the equivalence relation on the X_k's defined by $X_i \sim X_\iota$ if $dim_p(K_{i0} \cap K_{\iota 0}) = dim_p(K_{i0})$; that is the K_{i0} are "co-measurable". We will refer to the union of the elements of an equivalence class of this relation as an* ISLAND OF CO-MEASURABILITY*. For an island of co-measurability $X' = \bigcup_{i=t_1}^{t_2}(X_{i0} \setminus \cup_{j=1}^{n_i} X_{ij})$, with the additional condition that $dim_p(K_{ij}) = dim_p(K_{t_1 0})$ for all $t_1 \leq i \leq t_2$, $1 \leq j \leq n_i$ (i.e. all K_{ij}'s have the same p-dimension), we will call $K = \bigcap_{i,j} K_{ij}$ the* BUILDING BLOCK *for X'. Note that a building block is a p.p. definable subgroup.*

Remark 3.3.

1. *The equivalence relation is defined on the p.p. subgroups K_{i0}. This means that the parameters in the $\bar{y}$ variable play no role.*
2. *$K_{i0} \cap K_{j0}$ is a p.p. subgroup, therefore dim_p is defined on it.*
3. *By definition the intersection of two islands of co-measurability has lower dimension.*

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It will be convenient to extend the index function from subgroups to arbitrary (parametrically) definable sets in the following way:

Definition 3.4. *Let X be a definable set, K a p.p. subgroup. Define:*

$$|X : K| = \begin{cases} 0 & \textit{if } X \textit{ is empty} \\ t & \textit{if } t \textit{ is the least positive integer such that} \\ & \quad t \textit{ cosets of } K \textit{ cover } X \\ \infty & \textit{otherwise} \end{cases}$$

Remark 3.5. *If X' is an island of co-measurability, K its building block, then $|X' : K|$ will always be finite.*

Step one:

We assign values for an MS-measuring function h on a definable set X. A definable set X will be one of the following:

- *Empty set:* This is given MS-dimension 0 and MS-measure 0 (i.e. $h(\emptyset) = (0, 0)$).
- *Coset:* For X a non-empty coset of a p.p. subgroup K, define $h(X) = h_p(K)$.
- *A single disjunct:* Suppose $X = X_1 = X_{10} \setminus (X_{11} \cup .... \cup X_{1n_1})$. We can assume that none of the X_{1i}'s are empty. Suppose $K_{11}, ..., K_{1t}$ have finite index in K_{10}, and $K_{1(t+1)}, ..., K_{1n_1}$ have infinite index in K_{10}. Define:

$$dim(X) = dim(X_{10})$$
$$meas(X) = meas(X_{10}) + \Sigma_{\Delta \subseteq \{1,...,t\}} meas(\cap_{i\in\Delta} X_{1i})(-1)^{|\Delta|}$$

  Remark 3.6. *We know $meas(\cap_{i\in\Delta} X_{1i})$ because the intersection of two cosets $H + a$ and $G + b$ is either empty or a coset of $H \cap G$. As in this case H and G are both p.p. definable, $H \cap G$ will also be p.p. definable (Remark 2.7).*

- *Disjunctions:* Let $X = \cup_{i=1}^n X_i$ with $n > 1$, and suppose $max\{dim(X_i)\}_i = dim(X_1) = ... = dim(X_t)$, for suitable t. We define $h(X) = h(\cup_{i=1}^t X_i)$, essentially disregarding any disjunct of lower MS-dimension. For the following we can therefore assume that for a given definable set X the disjuncts all have the same MS-dimension.

  - *One island of co-measurability:*
    Suppose $X = \bigcup_i X_i = \bigcup_{i=1}^n (X_{i0} \setminus X_{i1} .... X_{in_i})$, and that it defines a single island of co-measurability (this would include an assumption that all the K_{i0} have the same dimension). Define:

$$dim(X) = dim_p(K_{10})$$
$$meas(X) = \Sigma_{\Delta \subseteq \{1,...,n\}} meas(\cap_{k\in\Delta} X_k)(-1)^{|\Delta|+1})$$

    Note: We have already defined $meas(\cap_{k\in\Delta} X_k)$ as $\cap_{k\in\Delta} X_k$ is a formula equivalent to a single disjunct formula(or the empty set).

    Remark 3.7. *If K is a building block for X then*

$$meas(X) = |X : K| meas_p(K)$$

  - *Finite disjunction of islands of co-measurability:* Suppose $X = \cup_{i=1}^m X^i$, where each X^i is an island of co-measurability. Recall we are assuming that $dim(X^1) = ... = dim(X^m)$ define

$$
\begin{aligned}
h(X) &= h(\bigcup_{i=0}^{t} X^i) \\
&= (d, \Sigma_{i=1}^{t} meas(X^i)).
\end{aligned}
$$

Remark 3.8. *This assignment is well defined. Suppose $\phi(\bar{x}, \bar{a})$ and $\psi(\bar{x}, \bar{b})$ define the same set. They will clearly contain the same islands of co-measurability. It is therefore sufficient to remark that the assignment is well defined for two different formulas defining the same island of co-measurability (note the building blocks given by two different formulas may be different). This is clear by index considerations on $K_1 \cap K_2$, and Remark 3.7, where K_1 and K_2 are the building blocks given by $\phi(\bar{x}, \bar{a})$ and $\psi(\bar{x}, \bar{b})$ respectively.*

Remark 3.9. *We may assume, by clause (c) and the way h is defined , that for a definable set X:*

1. *The X_{ij}'s are all non-empty (for all i, j).*
2. *$|X_{i0} : K_{ij}|$ are all finite (for all i) (i.e. $dim(X_{i0}) = dim(X_{ij})$).*
3. *$dim(X_{i0}) = dim(X_{10}) = dim(X_{ij})$.*

In the following we assume that X is of the form described in Remark 3.9.

Step two:
It is clear that (ii) and (iii) of Theorem 2.5 hold of this assignment.
Step three:
Firstly we need to show that for every definable set $\psi(x, \bar{y}) = \bigcup_{i=0}^{n}(\phi_{i0}(M, \bar{y}) \cap \bigcap_{j=1}^{n_i} \neg\phi_{ij}(M, \bar{y}))$ we have a finite $D_\psi \subseteq \mathbb{N} \times \mathbb{R}^{>0}$ such that $\forall \bar{a} \in M^m$, $h(\psi(x, \bar{a})) \in D_\psi$.

Remark 3.10. *We may reduce to considering formulas $\psi(x, \bar{y})$ such that $\psi(x, \bar{0})$ defines a single island of co-measurability. This is because the MS-dimension and MS-measure of a definable set is determined by the MS-dimension and MS-measure of its islands of co-measurability, and these are defined independently of parameters. So, we can assume all the disjunctions in $\psi(x, \bar{0})$ define "co-measurable" sets.*

Suppose $\phi(x, \bar{0})$ defines a single island of co-measurability, all the $\phi_{ij}(x, \bar{0})$ have the same MS-dimension, and therefore the corresponding building block $K = \bigcap_{i,j} \phi_{ij}(M, \bar{0})$ is such that $| \cup_{i=1}^{n} \phi_{i0}(M, \bar{0}) : K| = k_i$ (k_i finite). The possible values for $h(\psi(M, \bar{a}))$ are:

1. $(0, 0)$ if $M \models \neg\exists x \psi(x, \bar{a})$
2. $(dim(\phi_{00}(M, \bar{0})), m \times meas(K))$, where m is an integer, $1 \leq m < \Sigma_i k_i$.

This is because $\psi(M, \bar{a})$ is the union of up to $\Sigma_i k_i$ cosets of K, i.e., , $|\psi(M, \bar{a}) : K| = m \leq \Sigma_i k_i$, by Remark 3.9 $meas(\psi(M, \bar{a})) = m \times meas(K)$. This gives a finite number of possibilities for D_ψ.

Definability: We want to show that for any formula $\psi(x, \bar{y})$ and any $(d, \mu) \in D_\psi$ the set $\{\bar{y} : h(\psi(M, \bar{y})) = (d, \mu)\}$ is $\emptyset$-definable in M. By Theorem 2.5(i) we can disregard the case where $\psi(M, \bar{y})$ is empty (this case anyway is obviously definable by the formula $\neg\exists x\phi(x,\bar{y})$).

Case 1: $\psi(M,\bar{y})$ is p.p.

If $\psi(M,\bar{a})$ is non-empty, then $\psi(M,\bar{a})$ is a coset of $\psi(x,\bar{0})$, and $h(\psi(M,\bar{a})) = h(\psi(M,\bar{0}))$. Then $\{\bar{y} : h(\psi(M,\bar{y})) = h(\psi(M,\bar{0}))\}$ is defined by $\exists x(\psi(x,\bar{y}))$.

Case 2: $\psi(x,\bar{y}) = \phi_0(x,\bar{y}) \wedge (\bigwedge_{i=1}^{m} \neg\phi_i(x,\bar{y}))$

For $i = 1,...,m$, recall that $K_i = \{x \in M : M \models \phi_i(x,\bar{0})\}$ and $L = \bigcap_{i=1}^{m} K_i$. We may assume $|K_0 : K_i|$ is finite for all $i = 1,...,m$. So $|K_0 : L|$ is finite. For $\bar{a} \in M$ of appropriate length define $X_{\bar{a}}^i = \{x \in M : M \models \phi_i(x,\bar{a})\}$, and $X_{\bar{a}} = \bigcup_{i=1}^{m} X_{\bar{a}}^i$.

Recall that we are assuming $\psi(M,\bar{a})$ to be non-empty, and therefore $\phi_0(M,\bar{a})$ non-empty. For any $\bar{a} \in M$, $h(\psi(M,\bar{a}))$ will depend on $|X_{\bar{a}} : L|$. If we let $|X_{\bar{a}} : L| = k_{\bar{a}}$ and $h(L) = (d,\mu)$, then $h(\phi(M,\bar{a})) = (d,(k-k_{\bar{a}})\mu)$. We therefore need to find $\emptyset$-definable conditions on $\bar{y}$ for $|\bigcup_{i=1}^{m}\phi_i(M,\bar{y}) : L| = k_{\bar{y}}$.

We know from the "easy counting lemma" in [4] that as $X_{\bar{a}}^i \subseteq X_{\bar{a}}$, and $X_{\bar{a}}$ is the union of the $X_{\bar{a}}^i$ the following holds:

$$\Sigma_{\Delta\subseteq\{1,...,m\}}(-1)^{|\Delta|+1}|\cap_{i\in\Delta} X_{\bar{a}}^i : L| = |X_{\bar{a}} : K|$$

So it is enough to find $\emptyset$-definable conditions for determining $|\cap_{i\in\Delta} X_{\bar{a}}^i : L|$. This is determined by whether the intersection, $\cap_{i\in\Delta}X_{\bar{a}}^i$, is empty. That is to say if $\bar{a}$, $\bar{b}$ are such that both $\cap_{i\in\Delta}X_{\bar{a}}^i \neq \emptyset$ and $\cap_{i\in\Delta}X_{\bar{b}}^i \neq \emptyset$ then $|\cap_{i\in\Delta} X_{\bar{a}}^i : L| = |\cap_{i\in\Delta} X_{\bar{b}}^i : L|$. This is because they are both cosets of $\cap_{i\in\Delta}K_i$, a p.p. subgroup.

Clearly $\bigcap_{i\in\Delta} X_{\bar{y}}^i = \emptyset$ is a $\emptyset$-definable condition. So for any $t \in \mathbb{N}$ we have $\emptyset$-definable conditions determining when $|X_{\bar{y}} : L| = t$, hence $\emptyset$-definable conditions determining $h(\psi(M,\bar{y}))$.

Case 3: One island.

Remark 3.11. *First note that if a formula $\psi(x,\bar{a})$ with no redundant (i.e. empty) disjunctions has only one island of co-measurability then any "translate" of it $\psi(x,\bar{b})$ will have at most one island. This is because $\sim$ was defined independently of parameters. It therefore makes sense to consider definability for formulas with one island of co-measurability.*

Let $\psi(x,\bar{y})$ be such that $\psi(x,\bar{0})$ contains a single island of co-measurability, K_{ij} as above, and $X = \psi(x,\bar{a})$. Also suppose that the building block for this island is K, and $h(K) = (d,\mu)$. Recall that K will always have finite index in X, therefore, $h(X) = (d,|X : K|\cdot\mu)$. As above this is determined by which intersections of the X_{ij} are empty, which is an $\emptyset$-definable property. An explicit proof of this would work in exactly the same way as case 2.

Case 4: Finite disjunction of islands.

Suppose for some $\bar{a}$ our formula $\psi(x,\bar{a})$ contains at least two islands of co-measurability, and that these are defined by $\psi_i(x,\bar{a})$ for $i \in \{1,...,n\}$. Then for all $\bar{b} \in M^n$ and $i \neq j$, $dim(\psi_i(x,\bar{b}) \wedge \psi_j(x,\bar{b})) < max\{dim(\psi_i(x,\bar{b})), dim(\psi_i(x,\bar{b}))\}$ unless they are all empty. So the dimension and measure of $\psi(x,\bar{b})$ is determined by the dimension and measure of the non-empty $\psi_i(x,\bar{b})$. This is definable by

Case 3.

We give some examples to demonstrate how this we find $\phi(\bar{y})$ in some explicit cases:

Examples: 3.12. *a) Suppose that* $\psi(x,\bar{y}) = \phi_0(x,\bar{y}) \vee \phi_1(x,\bar{y})$*, where* $\phi_i(x,\bar{y})$ *are p.p., and that* $dim(\phi_0(x,\bar{0}) \wedge \phi_1(x,\bar{0})) = dim(\phi_0(x,\bar{0}))$*. Define* $K_i = \phi_i(M,\bar{0})$*. Suppose that* $h(K_i) = (d,\mu_i)$ *and that* $h(K_0 \cap K_1) = (d,\mu_2)$*. Let* $X_i = \phi_i M x, \bar{a})$*. Then* $X_0 \cup X_1$ *will have one of the following forms:*

		$\mathbf{h(X_0 \cup X_1)}$	$\emptyset-$***def formula***
1)	$X_0, X_1 = \emptyset$	$(0,0)$	$\neg\exists x\phi_0(x,\bar{y}) \wedge \neg\exists x\phi_1(x,\bar{y})$
2)	$X_0 = \emptyset, \quad X_1 \neq \emptyset$	(d,μ_1)	$\neg\exists x\phi_0(x,\bar{y}) \wedge \exists x\phi_1(x,\bar{y})$
3)	$X_0, X_1 \neq \emptyset,$ $X_1 \cap X_0 = \emptyset$	$(d,\mu_0+\mu_1)$	$\exists x\phi_0(x,\bar{y}) \wedge \exists x\phi_1(x,\bar{y})$ $\wedge\neg\exists x(\phi_0(x,\bar{y}) \wedge \phi_1(x,\bar{y}))$
4)	$X_0 \cap X_1 \neq \emptyset$	$(d,\mu_0+\mu_1-\mu_2)$	$\exists x(\phi_0(x,\bar{y}) \wedge \phi_1(x,\bar{y}))$

b) Take the formula $\psi(x,\bar{y}) = (\phi_{00}(x,\bar{y}) \wedge \neg\phi_{01}(x,\bar{y})) \vee \phi_{10}(x,\bar{y})$*, suppose this defines a single island with building block* K*. Let* $|K_{ij} : K| = l_{ij}$*,* $|K_{00} \cap K_{10} : K| = t$*, and let* $meas(K) = \mu$*. Then* $(X_{00} \backslash X_{01}) \cup X_{10}$ *will have one of the following forms:*

		$\mathbf{h(X)}$	$\emptyset-$***def formula***
1)	$X_{00}, X_{10} = \emptyset$	$(0,0)$	$\neg\exists x\phi_{00}(x,\bar{y}) \wedge \neg\exists x\phi_{10}(x,\bar{y})$
2)	$X_{00} \neq \emptyset$ $X_{10}, X_{01} = \emptyset$	$(d, l_{00}\mu)$	$\exists x\phi_{00}(x,\bar{y}) \wedge \neg\exists x\phi_{10}(x,\bar{y})$ $\wedge\neg\exists x\phi_{01}(x,\bar{y})$
3)	$X_{00}, X_{01} \neq \emptyset$ $X_{10} = \emptyset$	$(d,(l_{00}-l_{01})\mu)$	$\exists x\phi_{00}(x,\bar{y}) \wedge \neg\exists x\phi_{10}(x,\bar{y})$ $\wedge\exists x\phi_{01}(x,\bar{y})$
4)	$X_{00}, X_{10} \neq \emptyset, X_{01} = \emptyset$		*See 3.12 a)*
5)	$X_{00}, X_{01}, X_{10} \neq \emptyset$ $X_{10} \cap X_{01}, X_{10} \cap X_{00} = \emptyset$	$(d,(l_{00}-l_{01}$ $+l_{10})\mu)$	$\exists x\phi_{00}(x,\bar{y}) \wedge \exists x\phi_{10}(x,\bar{y})$ $\wedge\exists x\phi_{01}(x,\bar{y})$ $\wedge\neg\exists x(\phi_{10}(x,\bar{y}) \wedge \phi_{01}(x,\bar{y}))$ $\wedge\neg\exists x(\phi_{00}(x,\bar{y}) \wedge \phi_{10}(x,\bar{y}))$
6)	$X_{00}, X_{01}, X_{10}, X_{10} \cap X_{00} \neq \emptyset$ $X_{10} \cap X_{01} = \emptyset$	$(d,(l_{00}-l_{01}$ $+l_{10}-t)\mu)$	$\exists x(\phi_{00}(x,\bar{y}) \wedge \phi_{10}(x,\bar{y}))$ $\wedge\neg\exists x(\phi_{10}(x,\bar{y}) \wedge \phi_{01}(x,\bar{y}))$
7)	$X_{00}, X_{01}, X_{10} \neq \emptyset$ $X_{10} \cap X_{01}, X_{10} \cap X_{00} \neq \emptyset$	$(d,(l_{00}-l_{01}$ $+l_{10}+1)\mu)$	$\exists x(\phi_{10}(x,\bar{y}) \wedge \phi_{01}(x,\bar{y}))$

Step four:

It remains to show Theorem 2.5(iv). Define $\psi(\bar{x},\bar{a})$, K_{ij}, X_{ij}, X as above. Let $\pi : M^n \to M$ be the projection onto the i^{th} co-ordinate (note this is a p.p. definable function defined on M^n). The way condition (iv) of Theorem 2.5 is stated allows us to assume that all the fibres of π restricted to X have the same MS-dimension and MS-measure. Let $\pi(X) = Y$ so we have that $h(\pi^{-1}(y) \cap X) = (d, \mu)$, $\forall y \in Y$.

Lemma 3.13. *We can reduce to considering definable sets X with a single island of co-measurability.*

Proof of Lemma: Suppose that $X = X^1 \cup ... \cup X^n$, where X^i are islands of co-measurability. We assume that the desired result holds for single islands of co-measurability (i.e. the X^i), and show the the result holds for the whole set (i.e. X). We can therefore assume, by considering the definable set $\pi(X^i) \cap \pi(X^j)$ in $\pi(X^i)$, that the following holds for each i and j:

$$\begin{aligned} dim(\pi^{-1}(\pi(X^i) \cap \pi(X^j)) \cap X^i) &= dim(\pi(X^i) \cap \pi(X^j)) + dim(\pi^{-1}(y) \cap X^i) \\ meas(\pi^{-1}(\pi(X^i) \cap \pi(X^j)) \cap X^i) &= meas(\pi(X^i) \cap \pi(X^j))meas(\pi^{-1}(y) \cap X^i) \end{aligned} \tag{$\dagger$}$$

We may assume that the X^i are disjoint, and if the $\pi(X^i)$ are also all disjoint the result is clear. We assume that this is not the case. Let K_i be the building block of X^i and $Y_i = \pi(K_i)$. Note that as X^i is a finite union of cosets of K_i, $\pi(X^i)$ is a union of cosets of Y_i. If the $\pi(X^i)$ are not all disjoint, then there is some $i \neq j$ such that $\pi(X^i) \cap \pi(X^j) \neq \emptyset$. For simplicity we will assume that $i = 1$, $j = 2$ and that no three $\pi(X^i)$ intersect (the general case follows from this case). By a dimension argument $\pi(X^1) \cap \pi(X^2)$ is covered by finitely many translates of Y_1.

Claim: Suppose $\bigcup_{t=0}^{m}(Y_1 + a_t) \supseteq \pi(X^1) \cap \pi(X^2)$ is the sparsest covering of $\pi(X^1) \cap \pi(X^2)$ by cosets of Y_1, (i.e. each $(Y_1 + a_t) \cap (\pi(X^1) \cap \pi(X^2)) \neq \emptyset$), then $\pi(X^1) \cap \pi(X^2) = \bigcup_{t=0}^{m}(Y_1 + a_t)$.
Proof of Claim: Suppose $y \in \bigcup_{t=0}^{m}(Y_1 + a_t) \setminus (\pi(X^1) \cap \pi(X^2))$.

Note that as $\pi(X^1)$ is a union of cosets of Y_1, $(Y_1 + a_t) \subseteq \pi(X^1)$ for all $t = 0, ..., m$ so $y \in \pi(X)$. Without loss of generality let $y \in \pi(X^1) \setminus (\pi(X^1) \cap \pi(X^2))$. Therefore:

$$(dim, meas)(\pi^{-1}(y) \cap X) = (dim, meas)(\pi^{-1}(y) \cap X^1).$$

There is a coset $(Y_1 + a_t)$ which contains both y and a member y' of $\pi(X^1) \cap \pi(X^2)$. So,

$$(dim, meas)(\pi^{-1}(y') \cap X^1) \neq (dim, meas)(\pi^{-1}(y') \cap X).$$

By our assumptions that all fibres have the same dimension and measure in X we have:

$$(dim, meas)(\pi^{-1}(y) \cap X) = (dim, meas)(\pi^{-1}(y') \cap X).$$

Now as y, and y' are both in $(Y_1 + a_t)$ their fibres will intersect the same cosets of K_1. Therefore:

$$(dim, meas)(\pi^{-1}(y) \cap X^1) = (dim, meas)(\pi^{-1}(y') \cap X^1).$$

However, putting these together, we get:

$$\begin{aligned}(dim, meas)(\pi^{-1}(y) \cap X) &= (dim, meas)(\pi^{-1}(y) \cap X^1) \\ &= (dim, meas)(\pi^{-1}(y') \cap X^1) \\ &\neq (dim, meas)(\pi^{-1}(y') \cap X),\end{aligned}$$

which is a contradiction, yielding the claim.

Therefore $\pi(X^1) \cap \pi(X^2)$ is a union of cosets of both Y_1 and Y_2, i.e. $\pi(X^1) \cap \pi(X^2) = \bigcup_{t=0}^{m}(Y_1 + a_t) = \bigcup_{s=0}^{m}(Y_2 + b_s)$.

We can treat $\pi(X^1) \setminus \pi(X^2)$, $\pi(X^2) \setminus \pi(X^1)$ and $\pi(X^1) \cap \pi(X^2)$ as disjoint sets. It is therefore sufficient to show that we have the desired result for $y \in \pi(X^1) \cap \pi(X^2)$:

$$\begin{aligned}dim(\pi^{-1}(\pi(X^1) \cap \pi(X^2)) \cap X) &= dim(\pi(X^1) \cap \pi(X^2)) + dim(\pi^{-1}(y) \cap X) \\ meas(\pi^{-1}(\pi(X^1) \cap \pi(X^2)) \cap X) &= meas(\pi(X^1) \cap \pi(X^2))meas(\pi^{-1}(y) \cap X)\end{aligned}$$

Note that as X^1 and X^2 were assumed to be disjoint::

$$\begin{aligned}dim(\pi^{-1}(y) \cap X) &= dim(\pi^{-1}(y) \cap X^1) \\ meas(\pi^{-1}(y) \cap X) &= meas(\pi^{-1}(y) \cap X^1) + meas(\pi^{-1}(y) \cap X^2)\end{aligned}$$

So, using (†), we have that:

$$\begin{aligned}dim(\pi^{-1}(\pi(X^1) \cap \pi(X^2)) \cap X) &= dim(\pi^{-1}(\pi(X^1) \cap \pi(X^2)) \cap X^1) \\ &= dim(\pi(X^1) \cap \pi(X^2)) + dim(\pi^{-1}(y) \cap X^1) \\ &= dim(\pi(X^1) \cap \pi(X^2)) + dim(\pi^{-1}(y) \cap X)\end{aligned}$$

$$\begin{aligned}meas(\pi^{-1}(\pi(X^1) \cap \pi(X^2)) \cap X) &= meas(\pi^{-1}(\pi(X^1) \cap \pi(X^2)) \cap X^1) \\ &\quad + meas(\pi^{-1}(\pi(X^1) \cap \pi(X^2)) \cap X^2) \\ &= meas(\pi(X^1) \cap \pi(X^2))meas(\pi^{-1}(y) \cap X^1) \\ &\quad + meas(\pi(X^1) \cap \pi(X^2))meas(\pi^{-1}(y) \cap X^2) \\ &= meas(\pi(X^1) \cap \pi(X^2)) \\ &\quad (meas(\pi^{-1}(y) \cap X^1) + meas(\pi^{-1}(y) \cap X^2)) \\ &= meas(\pi(X^1) \cap \pi(X^2))meas(\pi^{-1}(y) \cap X)\end{aligned}$$

Hence as we can split X into similar disjoint pieces for all its islands we have shown that we can reduce to looking at single islands of co-measurability. We have not showed how to remove the assumption that no three images of island X^i intersect. This is tedious, but not difficult, and involves the same arguments as the above, with more complicated intersection considerations.

Now suppose we have a definable set X which is a single island of co-measurability. Let K be the building block for X. Consider the projection

π restricted to K, call this π', and let $\pi'(K) = Y'$ (a p.p. subgroup). Now as K, Y' and π' are all p.p. definable we have (by clause (b)):

$$(dim_p, meas_p)(K) = (dim_p(Y') + dim_p(ker(\pi')) \quad meas_p(Y')meas_p(ker(\pi')))$$

K has finite index in X and we may assume (by the way Theorem 2.5 is stated) fibres of π on X are of constant MS-dimension and MS-measure. Each fibre $\pi^{-1}(a) \cap X$, ($a \in Y$, where $Y = \pi(X)$) must be covered by a constant (finite) number, m say, of translations of $ker(\pi')$. We therefore have that $dim(\pi^{-1}(a) \cap X) = dim(ker(\pi'))$ and $meas(\pi^{-1}(a) \cap X) = m \times meas(ker(\pi'))$. Similarly, $|Y : Y'|$ must be finite (in fact $|Y : Y'| \leq |X : K|$), so $dim_p(Y') = dim(Y)$. Hence,

$$\begin{aligned} dim(X) &= dim_p(K) \\ &= dim_p(Y') + dim_p(ker(\pi')) \qquad \text{(by clause (b) in 3.1)} \\ &= dim(Y) + dim(\pi^{-1}(a) \cap X) \end{aligned}$$

Claim: Suppose $|Y : Y'| = n$ (i.e. $meas(Y) = meas(Y') \times n$). Then X is covered by mn cosets of K, where $m = |(\pi^{-1}(a) \cap X) : ker(\pi')|$.

Proof: Each of the m translates of $ker(\pi')$ that cover $(\pi^{-1}(a) \cap X)$ will map to the same coset of Y' in Y (i.e. the one that contains a, call this Y'_a). Also if x is contained in a translate of K whose intersection with $\pi^{-1}(a) \cap X$ is not empty (but x not necessarily in $\pi^{-1}(a) \cap X$), then x will also be mapped to Y'_a [a projection of a coset of K is either empty or a coset of the projection of K, i.e. Y']. There are n such cosets of Y'. So $m \times n$ translates cover X. Then:

$$\begin{aligned} meas(X) &= (mn)meas_p(K) \\ &= (mn)meas_p(Y')meas_p(ker(\pi')) \\ &= (n)meas_p(Y')(m)meas_p(ker(\pi')) \\ &= meas(Y)(meas(\pi^{-1}(a) \cap X)) \end{aligned}$$

So by Theorem 2.5 M is MS-measurable, using measure function h. The extension from h_p to h is clearly unique.

□

Remark 3.14. *We know (Lemma 3.3. in [3]) that if Y is a definable subgroup of an MS-measurable module M with MS-measuring function h, then:*

$$h(\bigcup_{i=1}^{n}(Y + a_i)) = (dim(Y), n \times meas(Y))$$

where $(Y + a_i)$ are disjoint cosets of Y. We therefore have that any MS-measuring function on modules fulfils clauses (a)-(c) of Theorem 3.1. From this we can conclude that a module M is MS-measurable if and only if it has such an MS-measuring function on p.p. definable sets (as described by clauses (a)-(a) in Theorem 3.1).

Corollary 3.15. *If M and N are both MS-measurable modules then so is $M \oplus N$.*

Proof: By 3.1 it is enough to look at the p.p. definable subgroups and functions of $M \oplus N$. We have measuring functions:

$$\begin{aligned} h_M : Def(M) \longmapsto \mathbb{N} \times \mathbb{R}^{>0} \cup \{(0,0)\} \\ h_N : Def(N) \longmapsto \mathbb{N} \times \mathbb{R}^{>0} \cup \{(0,0)\} \end{aligned}$$

Suppose $\phi(\bar{x})$ is a p.p. formula. The subset of $(M \oplus N)^m$ $\phi(\bar{x})$ defines will be of the form $M_\phi \oplus N_\phi$ where $M_\phi = \phi(M)$ and $N_\phi = \phi(N)$ are p.p. definable subsets of M^m and N^m respectively (See Chapter 2 of [7] for explanation). We need to define $h_p : Def_{p.p.}(M \oplus N) \mapsto \mathbb{N} \times \mathbb{R}^{>0} \cup \{(0,0)\}$. Suppose $h_M(M_\varphi) = (d_{M_\varphi}, m_{M_\varphi})$ and $h_N(N_\varphi) = (d_{N_\varphi}, m_{N_\varphi})$. Define:

$$h_p(M_\varphi \oplus N_\varphi) = (d_{M_\varphi} + d_{N_\varphi}, m_{M_\varphi} \cdot m_{N_\varphi})$$

It is obvious that this will fulfil clause (a) and (b) of Theorem 3.1. Let us therefore deal with (c). Suppose that X, Y are p.p. subsets of $M \oplus N$ defined by ψ and ϕ respectively (i.e. $X = M_\psi \oplus N_\psi$ and $Y = M_\phi \oplus N_\phi$). Firstly suppose $|X : Y| = n$, then $|M_\psi : M_\phi| = n_1$ and $|N_\psi : N_\phi| = n_2$ with $n = n_1 n_2$. So,

$$\begin{aligned} dim_p(X) &= dim_M(M_\psi) + dim_N(N_\psi) \\ &= dim_M(M_\phi) + dim_N(N_\phi) \\ &= dim_p(Y) \end{aligned}$$

Similarly $meas(X) = n \times meas(Y)$.

Secondly, suppose $|X : Y|$ is infinite, then either $|M_\psi : M_\phi|$ or $|N_\psi : N_\phi|$ is infinite. Therefore either $dim_M(M_\psi) > dim_M(M_\phi)$ or $dim_N(N_\psi) > dim_N(N_\phi)$, so we have that $dim_M(M_\psi) + dim_N(N_\psi) > dim_M(M_\phi) + dim_N(N_\phi)$, therefore $dim_p(X) > dim_p(Y)$.

So by Theorem 3.1 $M \oplus N$ will be MS-measurable.

$\square$

Corollary 3.16. *Let N be an MS-measurable module. Suppose M is a p.p. definable (without parameters) submodule of N^n. Then M, in the language of modules, is MS-measurable.*

Proof: Suppose N has MS-measuring function h_N. We may use this as an appropriate MS-measuring function on M. By Theorem 3.1 we need only check p.p. subgroups in M. Consider a p.p. formula $\phi(\bar{x})$ where $\bar{x} = x_1, ..., x_m$, this will define the same set in M as $\bigwedge_{i=1}^n \phi(y_1^i, ..., y_m^i) \bigwedge_{j=1}^m (y_j^1, ..., y_j^n \in M)$. This is a p.p. formula in N, so the result follows using the MS-measuring function $h_M(\phi(\bar{x})) = h_N(\bigwedge_{i=1}^n \phi(y_1^i, ..., y_m^i) \wedge \bigwedge_{j=1}^m (y_j^1, ..., y_j^n \in M))$.

4 Classification of Abelian Groups

In the following A will always be an Abelian group, $n \in \mathbb{N}$, p and q are prime. Also:

$\mathbb{Z}(n)$ will be the cyclic group with n elements
$\mathbb{Z}(p^\infty)$ will be the p-Prüfer group.
$\mathbb{Z}_{(p)}$ will be the p-adic integers
$\mathbb{Q}$ will be the rationals.

The theories of Abelian groups have been classified completely (see Fact 4.1). In this section we use this classification to determine which Abelian groups are MS-measurable and which are not.

Fact 4.1. *From Szmielew [8] we know that the complete theories of Abelian groups are of the form:*

$$Th(\bigoplus_{p\ prime} [\oplus_{n>0} \mathbb{Z}(p^n)^{\kappa_{(p,n)}} \oplus \mathbb{Z}(p^\infty)^{\lambda_p} \oplus \mathbb{Z}_{(p)}^{\mu_p}] \oplus \mathbb{Q}^\nu)$$

where $\kappa_{(p,n)}, \lambda_p, \mu_p$*, are cardinals* $\leq \omega$*,* $\nu \in \{0,1\}$*.*

Two abelian groups are elementarily equivalent if and only if they have the same Szmielew invariants (these are just the invariant sentences discussed in section two, Definition 2.9). In Abelian groups these have a particularly nice characterisation (see Appendix A.4., [5] or section 2.$\mathbb{Z}$ in [7] for details). We use this characterisation to check that certain ultraproducts of finite abelian groups are elementarily equivalent to particular infinite abelian groups.

The reader should note that from now on we will consider abelian groups up to elementary equivalence. We therefore use the notation for the group to denote its theory. For this chapter it will be convenient to write abelian groups as follows:

$$\oplus_{i \in I} \mathbb{Z}(p_i^{n_i})^{\kappa_i} \oplus_{j \in J} \mathbb{Z}(p_j^{m_j})^{\omega} \oplus_{k \in K} \mathbb{Z}(p_k^{\infty})^{\lambda_k} \oplus_{l \in L} \mathbb{Z}_{(p_l)}^{\mu_l} \oplus \mathbb{Q}^{\nu}$$

where κ_i, n_i, m_j are finite, λ_k, μ_l are cardinals $\leq \omega$, $\nu \in \{0,1\}$.

Remark 4.2. *We know from [6] that any Abelian group with infinite* U*-rank is not MS-measurable. The following groups are therefore not MS-measurable:*

$\mathbb{Z}(p^{\infty})^{\omega}$
$\mathbb{Z}_{(p)}^{\omega}$
$\oplus_{j \in J} \mathbb{Z}(p_j^{m_j})^{\omega}$*, where* J *infinite.*
$\oplus_{i \in I} \mathbb{Z}(p_i^{n_i})^{\kappa_i}$*, where some prime appears infinitely often,*
i.e. for some prime q*,* $\{i : p_i = q\}$ *is infinite.*

Note: Any direct sum of Abelian groups with any of the above will also have infinite rank.

Example 4.3. *We know [6] that any ultraproduct of finite cyclic groups is MS-measurable. Hence* $\mathbb{Q}$*,* $\mathbb{Z}(p^{\infty}) \oplus \mathbb{Z}_{(p)}$ *and* $\mathbb{Z}(p)^{\omega}$ *are all MS-measurable.*

Remark 4.4. *Any finite abelian group is MS-measurable. This is because any finite cyclic group is MS-measurable, just by letting the MS-dimension to be 0, and MS-measure to be size. We can then use Corollary 3.15 and the fundamental theorem of finite abelian groups.*

Proposition 4.5. *Suppose* $A = \oplus_{i \in I} \mathbb{Z}(p_i^{n_i})^{\kappa_i}$*, where* $\kappa_i < \omega$*,* $n_i < \omega$*,* $\{i : p_i = q\}$ *is finite for each prime* q*. Then* A *is MS-measurable.*

Proof: The case where I is finite is done by Remark 4.4. The case where I is infinite is done in appendix A. The proof involves showing that A is an ultraproduct of a one-dimensional asymptotic class. The way this is proved is very similar to the proof of Theorem 3.14 in [6].

□

Example 4.6. $\mathbb{Z}(p^n)^{\omega}$ *is MS-measurable. This is essentially shown by Elwes in [2].* $\mathbb{Z}(p^n)^{\omega}$ *is both* $\aleph_0$*-categorical and* ω*-stable, and therefore smoothly approximable (see p. 418, [6]). By 4.1. of [2]* $\mathbb{Z}(p^n)^{\omega}$ *is an ultraproduct of members of an* n*-dimensional asymptotic class, and therefore MS-measurable.*

Example 4.7. *$A = \oplus_{p \in P}(\mathbb{Z}(p^\infty) \oplus \mathbb{Z}_{(p)})$ is MS-measurable. If P is finite, this is obvious from Corollary 3.15 and Remark 4.3. If P is infinite A is still the ultraproduct of finite cyclic groups. First enumerate $P = \{p_1, p_2, ...\}$, where the p_i's are distinct primes. Let*

$$\mathcal{K}_i \quad = \quad \mathbb{Z}(p_1)^i \oplus \mathbb{Z}(p_2)^{i-1} \oplus ... \oplus \mathbb{Z}(p_i)$$

As all the p_i's are distinct each $\mathcal{K}_i$ is a cyclic group. We can see by checking Szmielew invariants that A is an ultraproduct of the $\mathcal{K}_i$'s.

Corollary 4.8. *Therefore by 3.15 and Examples 4.3-4.7 the following groups are MS-measurable:*

$$\oplus_{i \in I}\mathbb{Z}(p_i^{n_i})^{\kappa_i} \oplus \oplus_{j \in J}\mathbb{Z}(p_j^{m_j})^\omega \oplus \oplus_{k \in K}(\mathbb{Z}(p_k^\infty) \oplus \mathbb{Z}_{(p_k)})^{\mu_k} \oplus \mathbb{Q}^\nu$$

where n_i, κ_i, m_j, $\mu_k < \omega$, $\nu \in \{0, 1\}$, $|J|$ is finite, and I is such that for each prime q, $\{i : p_i = q\}$ is finite.

It remains to consider groups in which $\mathbb{Z}_{(p)}$ and $\mathbb{Z}(p^\infty)$ do not occur in pairs.

Example 4.9. *Consider the surjective function $\hat{p} : \mathbb{Z}(p^\infty) \to \mathbb{Z}(p^\infty)$ such that $\hat{p}(x) = px$. We have that $ker(\hat{p})$ is finite of size p. Suppose we had an MS-measuring function on $Z(p^\infty)$, then by clause (b) of Theorem 3.1 we would have $meas(\mathbb{Z}(p^\infty)) = p \times meas(\mathbb{Z}(p^\infty))$. This is clearly a contradiction.*

Remark 4.10. *If we now consider*

$$A = \oplus_{i \in I}\mathbb{Z}(p_i^{n_i})^{\kappa_i} \oplus \oplus_{j \in J}\mathbb{Z}(p_j^{m_j})^\omega \oplus \oplus_{k \in K}\mathbb{Z}({p_k}^\infty)^{\lambda_k} \oplus \mathbb{Q}^\nu$$

with $\kappa_i, m_j\lambda_k, |J| < \omega$ and $\nu \in \{0, 1\}$. We can use the function $f : x \to p_k x$ for some $k \in K$, and a similar counting argument to that in Example 4.9, to show that A is not MS-measurable.

Example 4.11. *In $\mathbb{Z}_{(p)}$ consider the function $\hat{p} : \mathbb{Z}_{(p)} \to p\mathbb{Z}_{(p)}$. This is a bijection so fibres have size one. Therefore by clause (b) of Theorem 3.1 (and Remark 3.14) if $\mathbb{Z}_{(p)}$ were MS-measurable we would have that $meas(\mathbb{Z}_{(p)}) = meas(p\mathbb{Z}_{(p)})$. However, we have that $|\mathbb{Z}_{(p)} : p\mathbb{Z}_{(p)}| = p$, so by clause (c) we would have that $meas(\mathbb{Z}_{(p)}) = p \cdot meas(p\mathbb{Z}_{(p)})$. This is clearly a contradiction, so $\mathbb{Z}_{(p)}$ is not MS-measurable.*

Remark 4.12. *We can use functions similar to those used in Examples 4.11 to show that the following Abelian groups are not MS-measurable:*

Group	*Function*
$\oplus_{i \in I}\mathbb{Z}(p_i^{n_i})^{\kappa_i} \oplus \oplus_{j \in J}\mathbb{Z}(p_j^{m_j})^\omega \oplus \oplus_{l \in L}\mathbb{Z}_{(p_l)}^{\mu_l} \oplus \mathbb{Q}^\nu$ $\mu_l, m_j, \kappa_i, \lvert J\rvert < \omega$ *and* $\nu \in \{0, 1\}$	$f : x \to p_l x$ *for some* $l \in L$
$\oplus_{i \in I}\mathbb{Z}(p_i^{n_i})^{\kappa_i} \oplus \oplus_{j \in J}\mathbb{Z}(p_j^{m_j})^\omega \oplus \oplus_{k \in K}\mathbb{Z}_{(p_k)}^{\lambda_k} \oplus \oplus_{l \in L}\mathbb{Z}(p_j^\infty)^{\mu_l} \oplus \mathbb{Q}^\nu$ *with* $\lambda_k, \mu_l, m_j, \kappa_i, \lvert J\rvert < \omega, \quad \nu \in \{0, 1\}$ *and either* $K \neq L$ *or for some* $l \in L, \quad \lambda_l \neq \mu_l$	$f : x \to p_l x$ *where* $l \notin L \cap K$ *or* $\lambda_l \neq \mu_l$

Theorem 4.13. *An abelian group A is MS-measurable, if and only if its theory is equivalent to that of an abelian group of the form:*

$$\oplus_{i \in I}\mathbb{Z}(p_i^{n_i})^{\kappa_i} \oplus \oplus_{j \in J}\mathbb{Z}(p_j^{m_j})^\omega \oplus \oplus_{k \in K}(\mathbb{Z}(p_k^\infty) \oplus \mathbb{Z}_{(p_k)})^{\mu_k} \oplus \mathbb{Q}^\nu$$

where n_i, κ_i, m_j, $\mu_k < \omega$, $\nu \in \{0, 1\}$, $|J|$ is finite, and I is such that for each prime q, $\{i : p_i = q\}$ is finite.

Proof: The right to left direction is Corollary 4.8. The left to right is seen by noticing all other cases are covered by Remark 4.2 and Remark 4.12.

Remark 4.14. *By inspection the MS-measurable abelian groups are precisely the pseudofinite abelian groups.*

5 $\mathbb{R}[t]$-valued measure

In [1] van den Dries and Cifu Lopes introduce a $\mathbb{Q}[t]$-valued measure on boolean combinations of cosets of $\mathbb{Z}^n$. The set up in this paper is rather different from ours. The idea is to start with a group Ω and consider the following sets (*Note:* In [1] Ω can be any group, here we will only be considering additive groups):

- $\mathcal{C}$ is a a set of subgroups of Ω, closed under $\cap$.
- $G = \{a + A : A \in \mathcal{C}, a \in \Omega\} \cup \{\emptyset\}$.
- A is the collection of finite unions of sets $X\backslash(Y_1 \cup ... \cup Y_m)$, where $X, Y_i \in \mathcal{G}$

Note: $\mathcal{C}$ can be any set of subgroups of Ω (i.e. it does not have to be all subgroups).

Given the form of definable sets in modules there is an obvious correspondence between these and the sets considered above. That is to say if we let M be a module, for every positive integer n we let $\mathcal{C}_n$ be the collection of p.p. definable subgroups of M^n, then $\mathcal{G}_n$ will be the set of p.p. cosets in M^n, and $\mathcal{A}_n$ will be all the definable subsets of M^n. We have the following comparison:

	Set-up in [1]	*Equivalent in our set up*
$\mathcal{C}$	set of subgroups of Ω closed under $\cap$	p.p. definable subgroups of M^n
$\mathcal{G}$	$\{a + A : A \in \mathcal{C}, a \in \Omega\} \cup \{\emptyset\}$	p.p. definable cosets of M^n
$\mathcal{A}$	The collection of finite unions of sets $X\backslash(Y_1 \cup ... \cup Y_m)$, $X, Y_i \in \mathcal{G}$ (i.e. the Boolean algebra on elements of $\mathcal{G}$)	Definable subsets of M^n

Definition 5.1. *A* MEASURE ON $\mathcal{A}$ *is a function $\mu : \mathcal{A} \to U$ (U an additive Abelian group) such that for all disjoint $X, Y \in \mathcal{A}$ we have $\mu(X \cup Y) = \mu(X) + \mu(Y)$.Observe that if μ is a measure we have $\mu(\emptyset) = 0$.*

Definition 5.2. *A measure μ is* LEFT INVARIANT *if for all $X \in \mathcal{C}$ and all $a \in \Omega$, $\mu(X) = \mu(a + X)$.*

Proposition 5.3. (Proposition 1.1 from [1]) *For n a positive integer, let $\mathcal{A}_n$ be the Boolean algebra on $\mathbb{Z}^n$ generated by cosets of p.p. subgroups of $\mathbb{Z}^n$. Then there is for each n a $\mathbb{Q}[t]$-valued measure (i.e. $U = \mathbb{Q}[t]$), μ_n on each $\mathcal{A}_n$ with the following properties:*

(*i*) $\mu_1(\{0\}) = 1$

(*ii*) $\mu_1(\mathbb{Z}^1) = t$

(*iii*) $\mu_n(X) = \mu_n(\bar{a} + X), \forall X \in \mathcal{A}_n, \forall \bar{a} \in \mathbb{Z}^n$

(*iv*) *if* $X \in \mathcal{A}_n,\ X \neq \emptyset$ *then* $\mu_n(X) \neq 0$

(*v*) $\mu_{n+m}(X \times Y) = \mu_n(X)\mu_m(Y)$

Note: The original proposition in [1] has an extra clause which is a direct consequence of (i) and (v), we therefore remove it.

This also being a measure on boolean combinations of cosets, it would seem plausible for there to be a correspondence between the MS-measuring function in modules and a $\mathbb{Q}[t]$-valued measure on definable subsets of modules. However, the Fubini property need not hold for a measure as in 5.3 (for example $\mathbb{Z}$ does not obey Fubini) so we will have to add an extra condition. In fact what we do is find a correspondence between the MS-measuring function in modules and a $\mathbb{R}[t]$-valued measure on definable subsets of modules. Essentially we have to do this as it could be that for some definable set X, $meas(X) \in \mathbb{R} \setminus \mathbb{Q}$.

Theorem 5.4. *Let $\mathcal{A}_n$ be the collection of definable sets in n variables of a module M, where n is a positive integer. Then M is MS-measurable if and only if for every $n > 0$, we have a measure $\mu_n : \mathcal{A}_n \to \mathbb{R}[t]$ such that the following properties hold for every $X \in \mathcal{A}_n$, $Y \in \mathcal{A}_m$ and $\bar{a} \in M^n$:*

(i) $\mu_n(\{\bar{a}\}) = 1$.

(ii) $\mu_n(X) = \mu_n(\bar{a} + X)$.

(iii) *If* $X \neq \emptyset$ *then* $\mu_n(X) \neq 0$.

(iv) $\mu_{n+m}(X \times Y) = \mu_n(X)\mu_m(Y)$.

(v) *(Fubini) Suppose $n \geq m$ and $f : X \to Y$ is a definable surjection, with constant fibre measure (i.e. there is an $l \in \mathbb{R}[t]$ with $Y = \{\bar{y} \in Y : \mu_{(n-m)}(f^{-1}(\bar{y})) = l\}$). Then $\mu_n(X) = l\mu_m(Y)$.*

Proof: In the proof we omit the subscripts on the measure to make our argument clearer. Then for $A \subseteq M^n$ we write $\mu(A)$ to mean $\mu_n(A)$, and $\mathcal{A}$ for the union of all $\mathcal{A}_n$.

($\Longleftarrow$) Suppose we have $\mu : \mathcal{A} \to \mathbb{R}[t]$ as described in the statement of the theorem. Consider a p.p. definable set X (*Note:* $\mu(X)$ is a polynomial in $\mathbb{R}[t]$). Define $dim_p(X)$ to be the degree (deg) of $\mu(X)$ and $meas_p(X)$ to be the modulus of the leading coefficient (lc) of $\mu(X)$. We use the modulus to guarantee that the MS-measure is positive.

Clause (a) of Theorem 3.1 is clear from condition (i) and additivity of measure. Clause (b) is clear, as by remark 2.8 and (ii) p.p. functions will have constant fibre measure (also, for non-zero polynomials p and q, $deg(pq) = deg(p) + deg(q)$ and $|lc(pq)| = |lc(p)||lc(q)|$).

For (c), suppose $X \supseteq Y$ are p.p. subgroups of $\mathcal{A}$. If $|X : Y| = k$ then we can find $\bar{a}_1, ..., \bar{a}_k$ of appropriate length such that $X = \bigcup_{i=1}^k (Y + \bar{a}_i)$, and the $(Y + \bar{a}_i)$ are disjoint. We have $\mu(Y + \bar{a}_i) = \mu(Y)$ so by Definition 5.1, $\mu(X) = \Sigma_{i=1}^k \mu(Y) = k\mu(Y)$. So $dim_p(X) = deg(\mu(X)) = deg(k\mu(Y)) = dim_p(Y)$ and $meas_p(X) = |lc(\mu(X))| = k \times |lc(\mu(Y))| = k \times meas_p(Y)$. If $|X : Y|$ is infinite, suppose for contradiction $deg(\mu(X)) = deg(\mu(Y)) = d$. For all $k \in \omega$ we can fine $\bar{a}_1, ..., \bar{a}_k$ such that, $X \supsetneq \bigcup_{i=1}^k (Y + \bar{a}_i)$, so $|lc(\mu(X))| \geq |k \times lc(\mu(Y))|$, clearly a contradiction.

So by Theorem 3.1 M will be measurable.

($\Longrightarrow$) Recall that in this set-up for every positive integer n, $\mathcal{C}_n$ is the set of p.p. definable subgroups of M^n. We make use of the following lemma from [1].

Lemma 5.5. *(Lemma 2.1 from [1]) Let $\mu_n : \mathcal{C}_n \to U$ (U an Abelian group) be a function such that $\mu_n(A) = d\mu_n(B)$ whenever $A, B \in \mathcal{C}_n$ and B is a subgroup of A of index d (finite). Then there is a unique left invariant U-valued measure on $\mathcal{A}_n$ extending μ_n.*

For X a p.p. definable subgroup of M^n define $\mu_n(X) = meas(X)t^{dim(X)}$. By Remark 3.14 we have that the condition in Lemma 5.5 is satisfied. We therefore have a suitable left invariant measure μ_n for every $\mathcal{A}_n$.

We need to show that this measure fulfils conditions (i)-(v). (i) is clear as $(dim, meas)(\{0\}) = (0, 1)$; (ii) is left invariance; (iii) is true because only the empty set has MS-measure $(0, 0)$.

(iv) essentially follows from additivity of measure, and the fact that we have this for p.p. cosets. If X and Y are both p.p. definable it is clear from our assignment that $\mu(X \times Y) = \mu(X)\mu(Y)$. For the case where X and Y are boolean combinations of p.p. cosets we use arguments that are similar to those in the proof of 3.1, and follow from additivity of measure. For simplicity and ease of understanding let us consider an example from which the general case is a clear extension. Let $X = X_0 \backslash X_1$ and $Y = (Y_0 \setminus Y_1) \cup (Y_2 \setminus Y_3)$ where $X_0, X_1, Y_0, Y_1, Y_2, Y_3$ are all cosets of p.p. subgroups, and $X_1 \subseteq X_0$, $Y_1 \subseteq Y_0$, $Y_3 \subseteq Y_2$. Using Remark 3.6 and the fact that we know this is a measure we get the following:

$$\mu(X) = \mu(X_0) - \mu(X_1)$$

$$\begin{aligned}\mu(Y) =& \mu(Y_0) + \mu(Y_2) - \mu(Y_0 \cap Y_2) - \mu(Y_1) - \mu(Y_3) \\ & + \mu(Y_0 \cap Y_3) + \mu(Y_2 \cap Y_1) - \mu(Y_1 \cap Y_3)\end{aligned}$$

$$\begin{aligned}X \times Y =& ((X_0 \times Y_0) \cup (X_0 \times Y_2)) \backslash \\ & ((X_1 \times Y_0) \cup (X_1 \times Y_2) \\ & \quad \cup ((X_0 \setminus X_1) \times (Y_1 \setminus (Y_1 \cap Y_2))) \\ & \quad \cup ((X_0 \setminus X_1) \times (Y_3 \setminus (Y_3 \cap Y_0))) \\ & \quad \cup ((X_0 \setminus X_1) \times (Y_1 \cap Y_3))) \\ \\ =& ((X_0 \times Y_0) \cup (X_0 \times Y_2)) \backslash \\ & ((X_1 \times Y_0) \cup (X_1 \times Y_2) \\ & \quad \cup ((X_0 \times Y_1) \setminus ((X_1 \times Y_1) \cup (X_0 \times (Y_2 \cap Y_1)))) \\ & \quad \cup ((X_0 \times Y_3) \setminus ((X_1 \times Y_3) \cup (X_0 \times (Y_0 \cap Y_3)))) \\ & \quad \cup ((X_0 \times (Y_1 \cap Y_3)) \setminus (X_1 \times (Y_1 \cap Y_3))))\end{aligned}$$

Now we can calculate:

$$\begin{aligned}\mu(X \times Y) =& \mu(X_0)\mu(Y_0) + \mu(X_0)\mu(Y_2) - \mu(X_0)\mu(Y_0 \cap Y_2)\\ &- (\mu(X_1)\mu(Y_0) + \mu(X_1)\mu(Y_2)) - \mu(X_1)\mu(Y_0 \cap Y_2)\\ &- (\mu(X_0)\mu(Y_1) - (\mu(X_1)\mu(Y_1) + \mu(X_0)\mu(Y_2 \cap Y_1) - \mu(X_1)\mu(Y_2 \cap Y_1)))\\ &- (\mu(X_0)\mu(Y_3) - (\mu(X_1)\mu(Y_3) + \mu(X_0)\mu(Y_0 \cap Y_3) - \mu(X_1)\mu(Y_0 \cap Y_3)))\\ &- (\mu(X_0)\mu(Y_1 \cap Y_3) - \mu(X_1)\mu(Y_1 \cap Y_3))\\ \\ =& \mu(X_0)(\mu(Y_0) + \mu(Y_2) - \mu(Y_0 \cap Y_2) - \mu(Y_1) - \mu(Y_3)\\ &+ \mu(Y_0 \cap Y_3) + \mu(Y_2 \cap Y_1) - \mu(Y_1 \cap Y_3))\\ &- \mu(X_1)(\mu(Y_0) + \mu(Y_2) - \mu(Y_0 \cap Y_2) - \mu(Y_1) - \mu(Y_3)\\ &+ \mu(Y_0 \cap Y_3) + \mu(Y_2 \cap Y_1) - \mu(Y_1 \cap Y_3))\\ \\ =& \mu(X)\mu(Y)\end{aligned}$$

It now remains to prove clause (v), the Fubini condition. We use the same notation used in Theorem 3.1. That is:

$$\begin{aligned}\psi(\bar{x}, \bar{y}) &= \bigvee_{i=1}^{n} (\phi_{i0}(\bar{x}, \bar{y}) \wedge \bigwedge_{j=1}^{n_i} \neg\phi_{ij}(\bar{x}, \bar{y})),\\ K_{ij} &= \{\bar{x} \in M : M \models \phi_{ij}(\bar{x}, \bar{0})\},\\ X_{ij} &= \{\bar{x} \in M : M \models \phi_{ij}(\bar{x}, \bar{a})\} \qquad \textit{Note: } X_{ij} \subsetneq X_{i0}\\ X_i &= X_{i0} \setminus (X_{i1} \cup ... \cup X_{in_i})\\ X &= \bigcup_{i=1}^{n} X_i = \bigcup_{i=1}^{n} (X_{i0} \setminus (X_{i1} \cup ... \cup X_{in_i}))\end{aligned}$$

Given a function $f : X \to Y$ definable, surjective with constant fibre measure, k, we show that $\mu(X) = k\mu(Y)$. We first outline the steps of this proof:

(1) *Step one:* reduce to considering co-ordinate projection.

(2) *Step two:* reduce to considering co-ordinate projections of definable sets of the form $X = X_{10} \setminus (X_{11} \cup ... \cup X_{1s})$ where the X_{1i} are cosets of p.p. subgroups.

(3) *Step three:* Assuming $X = X_0 \setminus (X_1 \cup ... \cup X_s)$, where X_i are cosets of p.p. subgroups. We split X into the following cases:

   (a) $dim(X_0) = dim(X_i)$ for all $1 \leq i \leq s$.

   (b) $dim(X_i) < dim(X_0)$ for all $1 \leq i \leq s$.

   We then show that in both cases we get the desired result.

Step one: Suppose $f : X \to Y$ is a definable surjective function, $X \subseteq M^n$, $Y \subseteq M^m$, consider $R = \{(\bar{x}, f(\bar{x})) : \bar{x} \in X\}$ and the projection π of R onto the last m coordinates. We have $\mu(R) = \mu(X)$ and $\mu(\pi^{-1}(y)) = \mu(f^{-1}(y))$ for

$y \in Y$. Therefore it is sufficient to get the desired result with π on R.

Step two: This is basically a consequence of additivity of measure and the fact that we can assume disjuncts to be disjoint. Let $X = \bigcup_{i=1}^{n} X_i$. We may assume that the X_i are disjoint (as intersections will be p.p. cosets) and that their images are either equal or disjoint (as the pre-image of a set is definable). Let $X_{l_i}, ..., X_{l_{(i+1)}-1}$ have the same image, so $\pi(X_{l_i} \cup ... \cup X_{l_{(i+1)}-1}) = \pi(X_{l_i})$. We therefore have a $t \in \mathbb{N}$ such that $\pi(X) = \cup_{i=1}^{t} \pi(X_{l_i})$. Suppose we know Fubini for each disjunct, i.e. that $\mu(X_i) = \mu(\pi(X_i))\mu(\pi^{-1}(y) \cap X_i)$ for some $y \in \pi(X_i)$, then by additivity and the fact that the X_i are disjoint:

$$\begin{aligned} \mu(X_{l_i} \cup ... \cup X_{l_{(i+1)}-1}) &= \mu(\pi(X_{l_i}))\mu(\pi^{-1}(y) \cap (X_{l_i} \cup ... \cup X_{l_{(i+1)}-1})) \\ &= \mu(\pi^{-1}(y) \cap X) \quad \text{for } y \in \pi(X_{l_i} \cup ... \cup X_{l_{(i+1)}-1}) \end{aligned}$$

Also as the fibres have constant measure we get, for $y \in \pi(X_{l_i} \cup ... \cup X_{l_{(i+1)}-1})$, and any $z \in \pi(X)$:

$$\mu(\pi^{-1}(y) \cap (X_{l_i} \cup ... \cup X_{l_{(i+1)}-1})) = \mu(\pi^{-1}(z) \cap X) \tag{$\dagger$}$$

So for $y_i \in \pi(X_{l_i} \cup ... \cup X_{l_{(i+1)}-1})$, and any $z \in \pi(X)$:

$$\begin{aligned} \mu(X) &= \sum_{i=1}^{t} \mu(X_{l_i} \cup ... \cup X_{l_{(i+1)}-1}) \\ &= \sum_{i=1}^{t} \mu(\pi(X_{l_i}))\mu(\pi^{-1}(y_i) \cap (X_{l_i} \cup ... \cup X_{l_{(i+1)}-1})) \\ &= \mu(\pi^{-1}(z) \cap X)(\sum_{i=1}^{t} \mu(\pi(X_{l_i})) && \text{by } (\dagger) \\ &= \mu(\pi^{-1}(z) \cap X)\mu(\pi(X)). \end{aligned}$$

Note: As we have reduced to considering a projection of X a single disjunct we do not need to worry about "islands of co-measurability".

Step three: From now on we will let $X = X_0 \setminus (X_1 \cup ... \cup X_s)$ (i.e. K_{ij} becomes K_j), π is a surjective projection $\pi : X \to Y$, with constant fibre measure k.

For case (a) where $dim(X_i) = dim(X_0)$ for all i we can use similar arguments to those used in proving Theorem 3.1. That is to say by considering indices on the set $K = \bigcap_{i=0}^{s} K_i$. If we let $\pi' : K \to Y'$ be π restricted to K, $|Y : Y'| = l$, and $|\pi^{-1}(x) : ker(\pi)| = m$. Then by arguments in 3.1 and the definition of μ:

$$\begin{aligned} \mu(X) &= ml \cdot meas(K)t^{dim(K)} \\ &= ml \cdot meas(ker(\pi'))meas(Y')t^{dim(ker(\pi'))+dim(Y')} \\ &= m \cdot meas(ker(\pi'))t^{dim(ker(\pi'))} l \cdot meas(Y')t^{dim(Y')} \\ &= k \cdot \mu(Y) \end{aligned}$$

Otherwise, for case (b), if $X = X_0 \setminus (X_1 \cup ... \cup X_s)$, then up to rearrangement of subscripts we can assume that for some $t \leq s$, $X_1, ..., X_t$ have finite index in X_0 and $X_{t+1}, ..., X_s$ have infinite index. X_0 will then be a disjoint union of cosets of $K_1 \cap ... \cap K_t$. If we get the required result on one of these cosets we can use additivity of measure to get (using similar arguments to those in step two) the required result on the whole of X. So it is sufficient to show case (b).

Therefore we may assume that all $X_1, ..., X_s$ have infinite index in X_0. We use the fact that as π is a projection it is defined on the whole of X_0 as well as X. Let π_0 be the projection such that $\pi_0|_X = \pi$, and let $Y_0 = \pi_0(X_0)$. For each $y_0 \in Y_0$ the fibre $\pi_0^{-1}(y_0)$ has the same measure, $k_0 = \mu(\pi_0^{-1}(y_0))$ say, as it is a coset of a p.p. subgroup (co-ordinate projection is p.p. definable).

All fibres of π have the same measure, therefore we can conclude that for any fibre F_0 of π_0, $X_1 \cup ... \cup X_s$ must either cover F_0 completely, or it must cover the same measured (proper) subset as it does in all other fibres of π_0 (i.e. those not completely covered by $X_1 \cup ... \cup X_s$). We may therefore assume that $X_1 \cup ... \cup X_s$ splits into the following sets (of which at least one is non-empty):

(1) $W = X_1 \cup ... \cup X_l$ such that each X_i is a union of fibres of π_0.

(2) $V = X_{l+1} \cup ... \cup X_s$ such that the intersection of V and each fibre of π_0 (not a subset of W) has constant measure.

First consider W, now π_0 is a projection, so all fibres of π_0 are cosets of a p.p. subgroup G_0. Consider π_0 restricted to X_i, where $1 \leq i \leq l$. Note that the fibres of π_0 restricted to X_i will still all be cosets of G_0. We obtain the following:

$$\begin{aligned}\mu(X_i) &= meas(X_i)t^{dim(X_i)} \\ &= meas(G_0)meas(\pi(X_i))t^{(dim(G_0)+dim(\pi(X_i)))} && \text{by MS-measurability} \\ &= \mu(G_0)\mu(\pi(X_i))\end{aligned}$$

So by additivity of measure we get that $\mu(W) = \mu(G_0)\mu(\pi(W))$.

Secondly, we may assume (again by additivity of measure arguments) V to be a single coset. Suppose $y \in Y$ (where $Y = \pi_0(X)$). Then $\pi_0^{-1}(y) \cap V \neq \emptyset$ (V meets each fibre of π_0 on X_0 with constant (non-zero) measure). Therefore $\pi_0(V) = Y$. Also for $y \in Y$, if we let $F_0 = \pi_0^{-1}(y)$ and $F_1 = \pi^{-1}(y)$, then the fibre of y in V is $F_0 \setminus F_1$ (i.e. $F_0 \setminus F_1 = (\pi_0^{-1}(y) \cap V)$). By MS-measurability and the fact that $(\pi_0^{-1}(y) \cap V)$ is a p.p. coset and has constant measure for all $y \in Y$ we see that:

$$\begin{aligned}meas(V) &= meas(\pi_0(V))meas(\pi_0^{-1}(y) \cap V) \\ dim(V) &= dim(\pi_0(V)) + dim(\pi_0^{-1}(y) \cap V)\end{aligned}$$

So we obtain the following:

$$\begin{aligned}\mu(V) &= meas(V)t^{dim(V)} \\ &= meas(\pi_0(V))meas(\pi_0^{-1}(y) \cap V)t^{(dim(\pi_0(V))+dim(\pi_0^{-1}(y)\cap V))} \\ &= meas(\pi_0(V))t^{dim(\pi_0(V))}meas(\pi_0^{-1}(y) \cap V)t^{dim(\pi_0^{-1}(y)\cap V)} \\ &= \mu(Y)\mu(F_0 \setminus F_1)\end{aligned}$$

Putting these together we calculate the following:

$$
\begin{aligned}
\mu(X) &= \mu(X_0) - \mu(W) - \mu(V) & \\
&= \mu(Y_0)\mu(G_0) - \mu(\pi(W))\mu(G_0) - \mu(V) & \\
&= (\mu(Y_0) - \mu(\pi(W)))\mu(G_0) - \mu(V) & \text{as } Y = Y_0 \setminus \pi(W) \\
&= \mu(Y)\mu(G_0) - \mu(V) & \text{as } Y = Y_0 \setminus \pi(W) \\
&= \mu(Y)\mu(G_0) - \mu(Y)\mu(F_0 \setminus F_1) & \\
&= \mu(Y)\mu(G_0) - \mu(Y)(\mu(F_0) - \mu(F_1)) & \\
&= \mu(Y)\mu(F_1) & \text{as } \mu(G_0) = \mu(F_0).
\end{aligned}
$$

Note: We have assumed the fibres on π to have constant measure, therefore they must all have the same measure as F_1. We have thereby shown the Fubini property.

□

Remark 5.6. *The Fubini in 5.4(v) on $\mathbb{R}(t)$-measured modules extends to to a full Fubini in the sense of Definition 2.2. This is seen by using similar arguments to those used in the proof of theorem 2.5 (Theorem 5.7 in [6]).*

A Appendix $\oplus_{i\in I}\mathbb{Z}(p_i^{n_i})^{k_i}$

In this appendix we complete the proof of Theorem 4.13 by proving Proposition 4.5. Recall that I is infinite subset of $\mathbb{N}$ (the case where I is finite is dealt with by Remark 4.4). We now investigate $A = \oplus_{i\in I}\mathbb{Z}(p_i^{n_i})^{k_i}$.

Assumptions A.1. *We can make the following assumptions:*

(1) A is written economically, i.e. if $i \neq j$ and $p_i = p_j$ then $n_i \neq n_j$.

(2) The p_i's are monotonically increasing, i.e. if $i < j$ then $p_i \leq p_j$.

(3) The n_i are finite (given by the classification).

(4) The k_i are finite. We can assume this as we have already covered other possibilities as follows:

*(a) Suppose there were a finite number of i's such that $k_i = \omega$.
Let $J = \{i : k_i = \omega\}$, we can write:*

$$A = \underbrace{\bigoplus_{i\in I\setminus J}\mathbb{Z}(p_i^{n_i})^{k_i}}_{A'} \quad \underbrace{\bigoplus_{i\in J}\mathbb{Z}(p_i^{n_i})^{\omega}}_{A''}.$$

Now as A'' is MS–measurable, A is MS–measurable if and only if A' is MS-measurable, by Corollary 3.15.

*(b) Suppose there were infinitely many i's such that $k_i = \omega$.
Again, let $J = \{i : k_i = \omega\}$ (J is now an infinite set). We get:*

$$A = \bigoplus_{i\in I\setminus J}\mathbb{Z}(p_i^{n_i})^{k_i} \bigoplus_{i\in J}\mathbb{Z}(p_i^{n_i})^{\omega}.$$

Suppose $J = \{j_1, j_2, ...\}$. Then we get the following infinite-index descending chain:

$$A \supseteq p_{j_1} A \supseteq p_{j_2} p_{j_1} A \supseteq ...$$

Therefore A has infinite U-rank and is not MS-measurable.

(5) *$\{i : p_i = q\}$ is finite for each prime q. Suppose not, i.e. $J = \{j : p_j = q\}$ is infinite for some q prime. We have:*

$$A = \underbrace{\bigoplus_{i \in I \setminus J} \mathbb{Z}(q^{n_i})^{k_i}}_{A'} \ \underbrace{\bigoplus_{i \in J} \mathbb{Z}(p_i^{n_i})^{k_i}}_{A''}.$$

If we consider $q^l A = \oplus_{i \in J} \mathbb{Z}(q^{n_i - l})^{k_i} \oplus A''$, (where we define $\mathbb{Z}(q^z) = \{0\}$ if $z < 0$) then $|q^l A : q^{l+1} A|$ is infinite. Therefore we have an infinite-index descending chain, $A \supseteq qA \supseteq q^2 A \supseteq ....$ Therefore A has infinite U–rank, so cannot be MS–measurable.

Theorem A.2. *Suppose*

$$A = \oplus_{i \in I} \mathbb{Z}(p_i^{n_i})^{k_i}$$

where:

(1) $k_i < \omega$

(2) $n_i < \omega$

(3) *$\{i : p_i = q\}$ is finite for each prime q.*

Then A is MS-measurable.

Proof: We show this by showing that A is the ultraproduct of a one-dimensional asymptotic class, therefore by Theorem 5.4 in [6] A is MS-measurable. Fixing A we consider the following finite groups (where $\iota \in I$):

$$K_\iota = \oplus_{i < \iota} \mathbb{Z}(p_i^{n_i})^{k_i}$$

Clearly $A = \prod_{\iota \in I} K_\iota / \mathcal{U}$. It is therefore sufficient to show that the K_ι form a one-dimensional asymptotic class. We make substantial use of the techniques used in Theorem 3.14 of [6]. We use the following definition of one-dimensional asymptotic class (Definition 1.2 in [6]).

Definition A.3. *Let $\mathcal{L}$ be a first order language, and $\mathcal{C}$ be a collection of finite $\mathcal{L}$-structures. Then $\mathcal{C}$ is a* 1-DIMENSIONAL ASYMPTOTIC CLASS *if the following hold for every $m \in \mathbb{N}$ and every formula $\phi(x, \bar{y})$, where $\bar{y} = (y_1, ..., y_m)$:*

(1) *There is a positive constant C and a finite set $E \subseteq \mathbb{R}^{>0}$ such that for every $M \in \mathcal{C}$ and $\bar{a} \in M^m$, either $|\phi(M, \bar{a})| \leq C$ or for some $\mu \in E$,*

$$||\phi(M, \bar{a})| - \mu |M|| \leq C |M|^{\frac{1}{2}}$$

(2) *For every $\mu \in E$, there is an $\mathcal{L}$-formula $\psi_\mu(\bar{y})$, such that, for all $M \in \mathcal{C}$. $\psi_\mu(M^m)$ is precisely the set of $\bar{a} \in M^m$ with*

$$||\phi(M, \bar{a})| - \mu|M|| \leq C|M|^{\frac{1}{2}}$$

For the following $\mathcal{L}$ will be the language of abelian groups. It is well known (see Appendix A.4., [5]) that in abelian groups p.p. formulas are of either of the form $p^n|t(\bar{x})$ or $t(\bar{x}) = 0$ where $t(\bar{x})$ is a term. The argument used here follows that of Theorem 3.14 in [6] very closely. The latter theorem establishes that a certain set of abelian groups (namely cyclic groups) forms a one-dimensional asymptotic class. They are therefore using the same language $\mathcal{L}$ as we are, and many of the same reductions hold here for exactly the same reason as they hold there. For more details see [6].

We may assume, by a similar inclusion-exclusion argument to that in Theorem 3.14 in [6], that the formula $\phi(x, \bar{y})$ is of the form:

$$\phi(x, \bar{y}) = (\wedge_{i=1}^{u}(\lambda_i x + y_i = 0) \wedge \wedge_{i=1}^{r}(p_i^{n_i}|l_i x + y_i)) \text{ where } \lambda_i, l_i \in \mathbb{Z}, p_i \text{ prime.}$$

We split the argument into sections. The first deals with $\phi(x, \bar{y})$ when it contains a conjunct $\lambda_i x + y_i = 0$, the second deals with $\phi(x, \bar{y})$ when it is a conjunction of terms $p_i^{n_i}|lx + y_j = 0$. The first section is relatively straight forward, the second is split into two cases.
Notation: For $a, b \in \mathbb{Z}$, $\gcd(a, b)$ denotes the greatest common denominator of a and b, $\text{lcm}(a, b)$ denotes the lowest common multiple of a and b.

1. $\phi(x, \bar{y})$ contains a conjunct $\lambda_i x + y_j = 0$

For simplicity we reduce the number of subscripts by dropping the i formula. $\mathbb{Z}(n)$ for positive integer n will denote the cyclic group of order n. From 3.14 in [6], we know the formula $\lambda x + y = 0$ holds in $\mathbb{Z}(n)$ if and only if $\gcd(l, n)|y$, moreover if this holds there will be exactly $\gcd(l, n)$ solutions.

Suppose the prime decomposition of λ is $\lambda = q_1^{m_1}...q_s^{m_s}$, where q_t's are prime and $q_t < q_{t+1}$. So for in the group $\mathbb{Z}(p_i^{n_i})^{k_i}$ we have the following possibilities:

(1) If $p_i = q_t$ for some t, then $\gcd(p_i^{n_i}, l) = \gcd(q_t^{n_i}, q_1^{m_1}...q_s^{m_s}) = q_t^{\min\{n_i, m_j\}} \leq q_t^{m_t}$. Therefore we have either:

- no solutions, if $q_t^{\min\{n_i, m_t\}} \not| y$.
- $(q_t^{\min\{n_i, m_t\}})^{k_i}$ solutions, if $q_t^{\min\{n_i, m_t\}}|y$.

Either way we have at most $(q_j^{\min\{n_i, m_j\}})^{k_i}$ solutions. Therefore at most $(q_t^{m_t})^{k_i}$ solutions.

(2) If $\gcd(p_i, \lambda) = 1$, then there is a unique solution to $\lambda x + y = 0$ in $\mathbb{Z}(p_i^{n_i})^{k_i}$.

Now let $J_j = \{i \in I : p_i = q_j\}$ and

$$\nu_j = \begin{cases} (\min J_j) - 1 & \text{for} \quad 1 \leq j \leq s \\ \nu_s + |J_s| & \text{for} \quad s + 1 \end{cases}$$

Assumption 5. from A.1 is important here as it guarantees that the J_j's are all finite. There are clearly only finitely many of them as λ has a finite factorisation.

We can now write A as follows (noting we are not changing the order in which the cyclic groups appear, and therefore not changing the class of K_ι's that we are considering):

$$A = \underbrace{\oplus_{i=1}^{\nu_1}\mathbb{Z}(p_i^{n_i})^{k_i}}_{\text{exactly one solution}} \; \underbrace{\oplus_{i\in J_1}\mathbb{Z}(q_1^{n_i})^{k_i}}_{\text{at most } \prod_{i\in J_1}(q_1^{m_1})^{k_i} \text{ solutions}} \; \underbrace{\oplus_{i=\nu_1+|J_1|}^{\nu_2}\mathbb{Z}(p_i^{n_i})^{k_i}}_{\text{exactly one solution}} \oplus \;... \; \underbrace{\oplus_{i>\nu_{s+1}}\mathbb{Z}(p_i^{n_i})^{k_i}}_{\text{exactly one solution}}$$

For large ι (i.e. $\iota > \nu_{s+1}$) the number of solutions of $\phi(x, \bar{y})$ in K_ι is at most $\prod_{j=1}^{s}\prod_{i\in J_j}(q_j^{m_j})^{k_i}$, this is also clearly true for $\iota \le \nu_{s+1}$. We choose $C = \prod_{j=1}^{s}\prod_{i\in J_j}(q_j^{m_j})^{k_i} + 1$, and we get that $|\phi(K_\iota, \bar{y})| < C$ for all ι. Clause 1. of Definition A.3 is therefore fulfilled in this case. Clause 2. is not relevant here.

2. $\phi(x, \bar{y})$ is $\wedge_{i=1}^{r}(p_i^{n_i}|l_i x + y_j)$

Below part (1) deals with formulas that contain only one prime, i.e. $p_i = p$ for all i. Part (2) then considers finite conjuncts of such formulas.

(1) Let $\phi(x, \bar{y}) = \wedge_{i=1}^{s} p^{n_i}|l_i x + y_i$. We know by 3.14 that in $\mathbb{Z}(q_j^{m_j})$ there are either:

   (a) No solutions.

   (b) $\frac{q_j^{m_j}}{L_j}$ solutions, where $L_j = \operatorname{lcm}\{\frac{e_{j1}}{d_{j1}}, ..., \frac{e_{js}}{d_{js}}\}$, $e_{ji} = \gcd(p^{n_i}, q_j^{m_j})$, $d_{ji} = \gcd(e_{ji}, l_i)$.

   Note: if $q_j \neq p$, $e_{ji} = 1$ for all i, therefore $L_j = 1$. So the solution set is the whole of $\mathbb{Z}(q_j^{m_j})$.

   Recall that $A = \oplus_{t\in I}\mathbb{Z}(p_t^{n_t})^{k_t}$, and $K_\iota = \oplus_{j<\iota}\mathbb{Z}(p_j^{n_j})^{k_j}$. Suppose k is such that $k = max\{j \in I : q_j = p\}$, recall that $\{j \in I : q_j = p\}$ is finite by Assumption A.1(5). Suppose that $\iota > k$ we have (when $\phi(K_\iota, \bar{y})$ non-empty):

$$\begin{aligned} |\phi(K_\iota \bar{y})| &= |\phi(\oplus_{\{j:q_j=p\}}\ \mathbb{Z}(q_j^{m_j})\bar{y})||\oplus_{\{j:q_j\neq p\}}\ \mathbb{Z}(q_j^{m_j})| \\ &= \prod_{\{j:q_j=p\}}\ (\tfrac{p^{m_j}}{L_j})^{k_j}|\oplus_{\{j:q_j\neq p\}}\ \mathbb{Z}(q_j^{m_j})| \end{aligned}$$

   Let $\mu = \prod_{\{j:q_j=p\}}\ (\frac{1}{L_j})^{k_j}$. For $\iota > k$, $\phi(K_\iota, \bar{y})$ non-empty we have:

$$||\phi(K_\iota, \bar{y})| - \mu|K_\iota|| = 0.$$

   Now if we choose $C = |K_k| + 1$ we get that for all ι

$$||\phi(K_\iota, \bar{y})| - |K_\iota|| < C.$$

   So Clause 1. of Definition A.3 is fulfilled.

(2) It is straight forward to go from part (1) to the general case. Formulas are finite, so there can only be finitely many different primes used in the formula. We can rewrite $\phi(x, \bar{y})$ as follows,

$$\phi(x, \bar{y}) = \wedge_{j=1}^{u}(\wedge_{i=1}^{s}(p_j^{n_{ij}}|l_{ij}x + y_{ij}))$$

Let $\phi_j(x, \bar{y}) = \wedge_{i=1}^{s}(p_j^{n_{ij}}|l_{ij}x + y_{ij})$. So $\phi(x, \bar{y}) = \wedge_{j=1}^{u}\phi_j(x, \bar{y})$. From part 1. we have, for each i, a k_i and a μ_i such that for any $\iota > k_i$ we have $||\phi_i(K_\iota, \bar{y})| - \mu|K_\iota|| = 0$. If we let $k = max\{k_1, ..., k_t\}$ and $\mu = \prod_{i=1}^{t} \mu_i$, then for $\iota > k$ we have (assuming $\phi(K_\iota, \bar{y})$ non-empty) $||\phi(K_\iota, \bar{y})| - \mu|K_\iota|| = 0$.

This is because the μ_i's are "independent". The number of solutions of $\phi(x, \bar{y})$ in K_ι is the product of the number of solutions in each summand. The number of solutions in each summand $\mathbb{Z}(q_l^{n_l})^{k_l}$ is only affected by subformula $\phi_i(x, \bar{y})$ if $q_l = p_i$ (i.e. if $q_l \neq p_i$, $\mathbb{Z}(q_l^{n_l})^{k_l} \models x = x \leftrightarrow \phi_i(x, \bar{y})$).

So if we choose $C = |K_k| + 1$ we get that for all ι

$$||\phi(K_\iota, \bar{y})| - |K_\iota|| < C.$$

It is therefore clear that clause 1. of Definition A.3 is fulfilled.

For the definability clause (clause 2. of Definition A.3) notice that μ and C work for any $\bar{y} \in M^m$, as long as $\phi(K_\iota, \bar{y})$ is not empty. Therefore we can use defining formula $\psi_\phi(\bar{y}) = \exists x\phi(x, \bar{y})$.

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