ADMISSIBILITY CONJECTURE AND KAZHDAN'S PROPERTY (T) FOR QUANTUM GROUPS

BISWARUP DAS, MATTHEW DAWS, AND PEKKA SALMI

Abstract. We give a partial solution to a long-standing open problem in the theory of quantum groups, namely we prove that all finite-dimensional representations of a wide class of locally compact quantum groups factor through matrix quantum groups (Admissibility Conjecture for quantum group representations). We use this to study Kazhdan's Property (T) for quantum groups with non-trivial scaling group, strengthening and generalising some of the earlier results obtained by Fima, Kyed and Sołtan, Chen and Ng, Daws, Skalski and Viselter, and Brannan and Kerr. Our main results are:

(i) All finite-dimensional unitary representations of locally compact quantum groups which are either unimodular or arise through a special bicrossed product construction are admissible.
(ii) A generalisation of a theorem of Wang which characterises Property (T) in terms of isolation of finite-dimensional irreducible representations in the spectrum.
(iii) A very short proof of the fact that quantum groups with Property (T) are unimodular.
(iv) A generalisation of a quantum version of a theorem of Bekka–Valette proven earlier for quantum groups with trivial scaling group, which characterises Property (T) in terms of non-existence of almost invariant vectors for weakly mixing representations.
(v) A generalisation of a quantum version of Kerr–Pichot theorem, proven earlier for quantum groups with trivial scaling group, which characterises Property (T) in terms of denseness properties of weakly mixing representations.

1. Introduction

Property (T) was introduced in the mid-1960s by Kazhdan, as a tool to demonstrate that a large class of lattices are finitely generated. The discovery of Property (T) was a cornerstone in group theory and the last decade saw its importance in many different subjects like ergodic theory, abstract harmonic analysis, operator algebras and some of the very recent topics like C*-tensor categories (see [8, 11, 37, 34] and references therein). In the late 1980s the subject of operator algebraic quantum groups gained prominence starting with the seminal work of Woronowicz [48], followed by works of Baaj, Skandalis, Woronowicz, Van Daele, Kustermans, Vaes and others [3, 46, 47, 27, 31]. Quantum groups can be looked upon as noncommutative analogues of locally compact groups, so quite naturally the notion of Property (T) appeared also in that more general context. Property (T) was first studied within the framework of Kac algebras (a precursor to the theory of locally compact quantum groups) [35], then for algebraic quantum groups [5] and discrete quantum groups [22, 29], and more recently for locally compact quantum groups [10, 18, 9].

By definition a locally compact group G has Property (T) if every unitary representation with approximately invariant vectors has in fact a non-zero invariant vector. This definition extends verbatim to locally compact *quantum* groups, using the natural extensions of the necessary terms. By a result of Fima, a discrete quantum group having Property (T) is necessarily a Kac algebra, which is equivalent to being unimodular in the case of discrete

Since $\mathbb{G}$ is unimodular, the universal modular automorphism groups of the right and left invariant weights are the same. It then follows from [26, Proposition 9.2] that

$$\Delta_u \circ \sigma_t^u = (\tau_t^u \otimes \sigma_t^u) \circ \Delta_u$$

and

$$\Delta_u \circ \sigma_t^u = (\sigma_t^u \otimes \tau_{-t}^u) \circ \Delta_u$$

for every $t \in \mathbb{R}$. Applying these to the matrix coefficient of V, we obtain

$$\Delta_u(\sigma_t^u(V_{ij})) = \sum_{k=1}^{n} \tau_t^u(V_{ik}) \otimes \sigma_t^u(V_{kj})$$

and

$$\Delta_u(\sigma_t^u(V_{ij})) = \sum_{k=1}^{n} \sigma_t^u(V_{ik}) \otimes \tau_{-t}^u(V_{kj}).$$

Then applying the counit of $C_0^u(\mathbb{G})$ to the above identities, it follows that $\tau_{-t}^u(\sigma_t^u(L_V)) \subset L_V$ and $\tau_t^u(\sigma_t^u(L_V)) \subset L_V$ for all $t \in \mathbb{R}$. This implies that for $X \in L_V$,

$$\tau_{2t}^u(X) = \left(\tau_t^u \circ \sigma_{-t}^u\right) \circ \left(\sigma_t^u \circ \tau_t^u\right)(X) \subset L_V.$$

Therefore, $\tau_t^u(L_V) \subset L_V$ for all $t \in \mathbb{R}$.

Let $\Lambda_{\mathbb{G}} : C_0^u(\mathbb{G}) \to C_0(\mathbb{G})$ be the reducing morphism. Then $\tau_t \circ \Lambda_{\mathbb{G}} = \Lambda_{\mathbb{G}} \circ \tau_t^u$ for all $t \in \mathbb{R}$ (see [26, Section 4]). Write $U = (U_{ij})$, and note that $\Lambda_{\mathbb{G}}(V_{ij}) = U_{ij}$. Since L_V is τ_t^u-invariant, it follows that L_U is τ_t-invariant. By Proposition 3.2, U is admissible. □

A class of examples of unimodular quantum groups with non-trivial scaling groups is given by Drinfeld's quantum doubles. Given a matched pair of locally compact quantum groups $\mathbb{G}$ and $\mathbb{H}$, the von Neumann algebra $L^\infty(\mathbb{G})\overline{\otimes}L^\infty(\mathbb{H})$ can be given the structure of a locally compact quantum group, called the double crossed product of $\mathbb{G}$ and $\mathbb{H}$ [4, Section 3 & Theorem 5.3]. If $\mathbb{H} = \widehat{\mathbb{G}}$, then a matching

$$m : L^\infty(\mathbb{G})\overline{\otimes}L^\infty(\widehat{\mathbb{G}}) \to L^\infty(\mathbb{G})\overline{\otimes}L^\infty(\widehat{\mathbb{G}})$$

is given by

$$m(X) = W_{\mathbb{G}^{\mathrm{op}}} X W_{\mathbb{G}^{\mathrm{op}}}^*,$$

where $W_{\mathbb{G}^{\mathrm{op}}}$ is the left multiplicative unitary of the opposite quantum group $\mathbb{G}^{\mathrm{op}}$ obtained from $\mathbb{G}$. The resulting double crossed product quantum group is precisely Drinfeld's quantum double, as has been shown in [4, Section 8]. Moreover, by [4, Proposition 8.1], Drinfeld's quantum doubles are always unimodular. It also follows from [4, Theorem 5.3] that if $\mathbb{G}$ has non-trivial scaling group, then the double crossed product has non-trivial scaling group as well. Thus Drinfeld's quantum double construction produces concrete examples of unimodular locally compact quantum groups with non-trivial scaling group, and by Theorem 4.3 the Admissibility Conjecture is true for such quantum groups.

5. Characterisation of Kazhdan's Property T for locally compact quantum groups

Let A be a C*-algebra and let $\pi : A \to B(H_\pi)$ and $\rho : A \to B(H_\rho)$ be two non-degenerate representations. The representation π is said to be *equivalent* to ρ if there exists a unitary $U : H_\pi \to H_\rho$ such that $U^*\rho(x)U = \pi(x)$ for every $x \in A$. If U is only an isometry, then we will say that π is *contained* in ρ and write $\pi \subset \rho$ (in other words, π is a *subrepresentation* of ρ). Now let $\mathcal{S}$ be a set of representations of A. We say that the representation π is *weakly* *contained* in $\mathcal{S}$ and write $\pi \prec \mathcal{S}$ if $\bigcap_{\rho\in\mathcal{S}} \operatorname{Ker}\rho \subset \operatorname{Ker}\pi$. We will adopt the convention that whenever $\pi \prec \{\rho\}$ we will simply write $\pi \prec \rho$.

Let $\widehat{A}$ denote the set of inequivalent irreducible representations of A. The *closure* of $\mathcal{S} \subset \widehat{A}$ is defined as

$$\overline{\mathcal{S}} = \{\, \pi \in \widehat{A} : \pi \prec \mathcal{S} \,\}.$$

From [21, Lemma 1.6] it follows that the above closure defines a topology on $\widehat{A}$, which is referred to as the Fell topology on $\widehat{A}$ (in [21] this was called the hull-kernel topology on $\widehat{A}$). The following result from [10, Lemma 2.1] is crucial in the sequel.

Proposition 5.1. *Let A be a C*-algebra. If $\rho \in \widehat{A}$ is finite-dimensional, then $\{\rho\}$ is a closed subset of $\widehat{A}$. Moreover, the following statements are equivalent for $\rho \in \widehat{A}$:*

(i) *ρ is an isolated point in $\widehat{A}$.*
(ii) *If a representation π of A satisfies $\rho \prec \pi$, then $\rho \subset \pi$.*
(iii) *$A = \operatorname{Ker}\rho \oplus \bigcap_{\nu\in\widehat{A}\setminus\{\rho\}} \operatorname{Ker}\nu$.*

Remark 5.2. It is worthwhile to mention that recently in [9, Definition 3.3] the authors have used a slightly different notion of weak containment, namely a C*-algebraic version of [50, Definition 7.3.5]. However, we will be concerned with irreducible representations in which case the definition coincide (as mentioned after Definition 3.3 in [9]).

The following definition is a natural extension of containment to the setting of unitary representations of locally compact quantum groups (see [16, Definition 3.2]).

Definition 5.3. Let $U \in M(C_0(\mathbb{G}) \otimes \mathcal{K}(H))$ and $V \in M(C_0(\mathbb{G}) \otimes \mathcal{K}(K))$ be two unitary representations of a locally compact quantum group $\mathbb{G}$, and let $\pi_U : C_0^u(\widehat{\mathbb{G}}) \to B(H)$ and $\pi_V : C_0^u(\widehat{\mathbb{G}}) \to B(K)$ be the associated representations of $C_0^u(\widehat{\mathbb{G}})$ (i.e. $U = (\iota \otimes \pi_U)(\mathbb{W})$ and $V = (\iota \otimes \pi_V)(\mathbb{W})$). If $\pi_U \prec \pi_V$, we say U is *weakly contained* in V and write $U \prec V$. If $\pi_U \subset \pi_V$, we say that U is *contained* in V (or that U is a *subrepresentation* of V) and write $U \subset V$.

Remark 5.4. Let $U \prec V$, so that $\pi_U \prec \pi_V$. Let W be a corepresentation. We claim that $W \textcircled{\scriptsize T} U \prec W \textcircled{\scriptsize T} V$. Indeed, this is equivalent to $\pi_{W \textcircled{\tiny T} U} = (\pi_W \otimes \pi_U) \circ \chi \circ \widehat{\Delta}_u \prec (\pi_W \otimes \pi_V) \circ \chi \circ \widehat{\Delta}_u = \pi_{W \textcircled{\tiny T} V}$, which by definition, is equivalent to showing that $\ker(\pi_W \otimes \pi_V) \circ \widehat{\Delta}_u \subseteq \ker(\pi_W \otimes \pi_U) \circ \widehat{\Delta}_u$.

Let W act on H_W, and let $a \in \ker(\pi_W \otimes \pi_V) \circ \widehat{\Delta}_u$. For $\omega \in \mathcal{K}(H_W)^*$ let $b = (\omega \circ \pi_W \otimes \iota)\widehat{\Delta}_u(a)$, so that $\pi_V(b) = 0$. As $\pi_U \prec \pi_V$, it follows that $\pi_U(b) = 0$, so that $(\omega \otimes \iota)(\pi_W \otimes \pi_U) \circ \widehat{\Delta}_u(a) = 0$. As ω was arbitrary, $(\pi_W \otimes \pi_U) \circ \widehat{\Delta}_u(a) = 0$, as required.

We next recall the definitions of invariant and almost invariant vectors for quantum group representations (see [16, Definition 3.2]).

Definition 5.5. Let $U \in M(C_0(\mathbb{G}) \otimes \mathcal{K}(H))$ be a representation of a locally compact quantum group $\mathbb{G}$. A vector $\xi \in H$ is called *invariant* for U if $U(\eta \otimes \xi) = \eta \otimes \xi$ for all $\eta \in L^2(\mathbb{G})$. U is said to have *almost invariant vectors* if there exits a net $(\xi_\alpha)_\alpha$ of unit vectors in H such that $\|U(\eta \otimes \xi_\alpha) - \eta \otimes \xi_\alpha\| \to 0$ for all $\eta \in L^2(\mathbb{G})$.

Note that if U and V are similar representations of a locally compact quantum group $\mathbb{G}$, then U has an invariant vector if and only if V has. The analogous statement holds for almost invariant vectors.

We will need invariant means in the following arguments, so next we set up the necessary terminology for those.

Definition 5.6. The (reduced) *Fourier–Stieltjes algebra* of $\mathbb{G}$ is defined by

$$B(\mathbb{G}) = \operatorname{span}\{\, (\iota \otimes \omega)(\mathbb{W}) : \omega \in C_0^u(\widehat{\mathbb{G}})^* \,\}.$$

Then the *Eberlein algebra* of $\mathbb{G}$ is defined by

$$E(\mathbb{G}) = \overline{B(\mathbb{G})}^{\|\cdot\|_{B(L^2(\mathbb{G}))}}.$$

Note that $B(\mathbb{G}) \subset M(C_0(\mathbb{G}))$ and that $B(\mathbb{G})$ is a subalgebra of $M(C_0(\mathbb{G}))$. It then follows that $E(\mathbb{G})$ is a closed subalgebra of $M(C_0(\mathbb{G}))$. When $\mathbb{G}$ is of Kac type, $E(\mathbb{G})$ is self-adjoint and so a C*-subalgebra of $M(C_0(\mathbb{G}))$ (see for example [14, Section 7]).

The Banach algebra $L^1(\mathbb{G})$ acts on its dual $L^\infty(\mathbb{G})$ by

$$x \cdot \omega = (\omega \otimes \iota)(\Delta(x)), \qquad \omega \in L^1(\mathbb{G}), x \in L^\infty(\mathbb{G}),$$

and

$$\omega \cdot x = (\iota \otimes \omega)(\Delta(x)), \qquad \omega \in L^1(\mathbb{G}), x \in L^\infty(\mathbb{G}),$$

making $L^\infty(\mathbb{G})$ an $L^1(\mathbb{G})$-bimodule. It is easy to see that $E(\mathbb{G})$ is invariant under these actions. This allows us to define the notion of an invariant mean on $E(\mathbb{G})$. It follows from [18, Proposition 3.15] that $E(\mathbb{G})$ admits an *invariant mean* $\mu \in E(\mathbb{G})^*$ in the sense that $\|\mu\| = \mu(1) = 1$ and

$$\mu(\omega \cdot x) = \omega(1)\mu(x) = \mu(x \cdot \omega) \qquad \omega \in L^1(\mathbb{G}), x \in E(\mathbb{G}).$$

Note that the Hopf $*$-algebra $\mathcal{AP}(\mathbb{G})$ underlying the quantum Bohr compactification is contained in $B(\mathbb{G})$, and therefore $AP(\mathbb{G}) \subset E(\mathbb{G})$. The uniqueness of the Haar state of a compact quantum group implies the following result.

Lemma 5.7. *Let M be the restriction of the invariant mean $\mu \in E(\mathbb{G})^*$ to $AP(\mathbb{G})$. Then M is the Haar state of the compact quantum group $(AP(\mathbb{G}), \Delta)$.*

Lemma 5.8. *Let $X \in B(L^2(\mathbb{G}))\overline{\otimes}B(H)$ for some Hilbert space H, and suppose that $(\iota \otimes \omega)(X) \in E(\mathbb{G})$ for all $\omega \in B(H)_*$. Let $\nu \in E(\mathbb{G})^*$. Then there exists an operator $T \in B(H)$ such that for all $\omega \in B(H)_*$*

$$\langle T, \omega \rangle = \langle \nu, (\iota \otimes \omega)(X) \rangle.$$

We will be denoting this operator by $(\nu \otimes \iota)(X)$, so that

$$\omega\big((\nu \otimes \iota)(X)\big) = \nu\big((\iota \otimes \omega)(X)\big)$$

for all $\omega \in B(H)_*$.

Proof. The map

$$\omega \mapsto \langle \nu, (\iota \otimes \omega)(X) \rangle : B(H)_* \to \mathbb{C}$$

defines a bounded functional on $B(H)_*$, which gives the existence of the operator $T \in B(H)$. $\square$

The next result gives a formula for the orthogonal projection onto the set of invariant vectors for a unitary corepresentation of $C_0(\mathbb{G})$ (also see [18, Proposition 3.14]).

Lemma 5.9. *Let $V \in M(C_0(\mathbb{G}) \otimes \mathcal{K}(H))$ be a unitary representation of $\mathbb{G}$ and let $\mu \in E(\mathbb{G})^*$ be the invariant mean on $E(\mathbb{G})$. The operator $P = (\mu \otimes \iota)(V) \in B(H)$ is a projection onto the subspace of invariant vectors for V.*

Proof. Due to the subtle definition of $(\mu \otimes \iota)(V)$, we include a careful calculation of the fact that the image of P consists of invariant vectors. Given $\sigma \in L^1(\mathbb{G})$ and $\omega \in B(H)_*$, we have

$$(\sigma \otimes \omega)\big(V(1 \otimes P)\big) = \omega((\sigma \otimes \iota)(V)P) = \mu\big((\sigma \otimes \iota \otimes \omega)(V_{13}V_{23})\big)$$

due to the commutation relation in Lemma 5.8. Continuing from here using the fact that V is a representation, we have

$$\begin{aligned}(\sigma \otimes \omega)\big(V(1 \otimes P)\big) &= \mu\big((\sigma \otimes \iota \otimes \omega)((\Delta \otimes \iota)(V))\big) = \mu\big((\sigma \otimes \iota)(\Delta((\iota \otimes \omega)(V)))\big) \\ &= \sigma(1)\mu((\iota \otimes \omega)(V)) = (\sigma \otimes \omega)(1 \otimes P)\end{aligned}$$

due to the invariance of μ. It follows that the image of P consists of invariant vectors. That P is an idempotent map is also easy to compute. Notice that $\|P\| \leq 1$, because $\|\mu\| = 1$, from which it follows that P is the orthogonal projection onto the subspace of invariant vectors. $\square$

The following result from [16, Corollary 2.5 and Corollary 2.8] gives a nice criterion for the existence of invariant and almost invariant vectors. Denote the trivial representation of $\mathbb{G}$ by 1.

Proposition 5.10. *Let $U \in M(C_0(\mathbb{G}) \otimes \mathcal{K}(H))$ be a unitary representation of a locally quantum group $\mathbb{G}$.*

(i) *U has a nonzero invariant vector if and only if $1 \subset U$.*
(ii) *U has almost invariant vectors if and only if $1 \prec U$.*

We now recall the definition of Kazhdan's Property (T) for quantum groups (see [10, Definition 3.1], which goes back to [22, Definition 6]).

Definition 5.11. A locally compact quantum group $\mathbb{G}$ has (*Kazhdan's*) *Property (T)* if every unitary representation of $\mathbb{G}$ that has almost invariant vectors has a nonzero invariant vector.

It follows from Proposition 5.10 that $\mathbb{G}$ has Property (T) if and only if for every unitary representation U of $\mathbb{G}$

$$1 \prec U \implies 1 \subset U.$$

The following theorem [10, Proposition 3.2 and Theorem 3.6] and [9, Theorem 4.7] gives a series of equivalent conditions to Property (T).

Theorem 5.12. *Let $\mathbb{G}$ be a locally compact quantum group. The following statements are equivalent:*

(T1) *$\mathbb{G}$ has Property (T).*
(T2) *The counit $\widehat{\epsilon}_u$ is an isolated point in $\widehat{C_0^u(\widehat{\mathbb{G}})}$.*
(T3) $C_0^u(\widehat{\mathbb{G}}) = \operatorname{Ker} \widehat{\epsilon}_u \oplus \mathbb{C}$.
(T4) *There is a projection $p \in M(C_0^u(\widehat{\mathbb{G}}))$ such that $pC_0^u(\widehat{\mathbb{G}})p = \mathbb{C}p$ and $\widehat{\epsilon}_u(p) = 1$.*

If $\mathbb{G}$ has trivial scaling group, then the above conditions are further equivalent to the following (quantum version of Wang's theorem [44, Theorem 2.1]*):*

(T5) *Every finite-dimensional irreducible representation of $C_0^u(\widehat{\mathbb{G}})$ is an isolated point in $\widehat{C_0^u(\widehat{\mathbb{G}})}$.*
(T6) *$C_0^u(\widehat{\mathbb{G}}) \cong B \oplus M_n(\mathbb{C})$ for some C^*-algebra B and $n \in \mathbb{N}_0$.*

We will prove that for a locally compact quantum group with non-trivial scaling group, (T5) as well as a suitable generalisation of (T6) are equivalent with (T1)–(T4).

We first make two observations, which will be used later (also see [43, Proposition 3.14 & Proposition 3.15]).

Lemma 5.13. *Let $U \in M(C_0(\mathbb{G}) \otimes M_n(\mathbb{C}))$ be a finite-dimensional admissible unitary representation of $\mathbb{G}$. Then $1 \subset U^c \textcircled{\scriptsize T} U$.*

Proof. Write $U = (U_{ij})_{i,j=1}^n$, and define $\overline{U} = (U_{ij}^*)_{i,j=1}^n$. Since U is admissible, $\overline{U} \in E(\mathbb{G}) \otimes M_n(\mathbb{C})$. Define $V = \overline{U} \textcircled{\scriptsize T} U \in E(\mathbb{G}) \otimes (M_n(\mathbb{C}) \otimes M_n(\mathbb{C}))$. Let μ be the invariant mean on $E(\mathbb{G})$. We will show that $(\mu \otimes \iota)(V) \neq 0$, where we are using the notation of Lemma 5.8. Towards a contradiction, suppose that $(\mu \otimes \iota)(V) = 0$, so that for all $\nu \in (M_n(\mathbb{C}) \otimes M_n(\mathbb{C}))_*$ we have $\mu((\iota \otimes \nu)(V)) = 0$. Choosing ν such that $(\iota \otimes \nu)(V) = U_{ij}^* U_{ij}$ leads to $\mu(U_{ij}^* U_{ij}) = 0$ for all $i, j = 1, 2, \ldots, n$. Therefore,

$$\mu\Big(\sum_{i=1}^n U_{ij}^* U_{ij}\Big) = \mu(1) = 0,$$

which cannot happen as $\mu(1) = 1$.

By Lemma 5.9, any vector in the image of $(\mu \otimes \iota)(V)$ is an invariant vector for V. Since $\overline{U}$ is similar to U^c (as U is admissible, Remark 3.5), the representation V is similar to $U^c \textcircled{\scriptsize T} U$ and the result follows from Proposition 5.10-(i). □

The following result may be considered as an extension of [10, Proposition 3.5] (see also [8, Proposition A.1.12] and [29, Theorem 2.6]).

Theorem 5.14. *Let $U \in M(C_0(\mathbb{G}) \otimes M_n(\mathbb{C}))$ and $V \in M(C_0(\mathbb{G}) \otimes \mathcal{K}(H))$ be unitary representations of a locally compact quantum group $\mathbb{G}$ with U being admissible. Then the following are equivalent:*

(i) *The representations U and V contain a common finite-dimensional unitary representation.*
(ii) *The representation $U^c \textcircled{\scriptsize T} V$ contains the trivial representation.*

Proof. (i) $\Longrightarrow$ (ii): Let W be a common finite-dimensional unitary representation of U and V. Then W is also admissible as U is admissible. We have $W^c \textcircled{\scriptsize T} W \subset U^c \textcircled{\scriptsize T} V$ and by Lemma 5.13, $1 \subset W^c \textcircled{\scriptsize T} W$.

(ii) $\Longrightarrow$ (i): Let μ denote the invariant mean on $E(\mathbb{G})$. Let $x \in B(H, \mathbb{C}^n)$ and $y = (\mu \otimes \iota)(U^*(1 \otimes x)V)$ so that $y \in B(H, \mathbb{C}^n)$. Now

$$\begin{aligned} U^*(1 \otimes y)V &= (\mu \otimes \iota \otimes \iota)\big(U_{23}^* U_{13}^* (1 \otimes 1 \otimes x) V_{13} V_{23}\big) \\ &= (\mu \otimes \iota \otimes \iota) \circ (\Delta \otimes \iota)\big(U^*(1 \otimes x)V\big) = 1 \otimes y \end{aligned}$$

due to the invariance of μ (see the proof of Lemma 5.9 for making the above calculation more rigorous). Since U is unitary, we have

$$U(1 \otimes y) = (1 \otimes y)V.$$

Next we show that for some x, the resulting y is nonzero,

It follows from the hypothesis, via Proposition 5.10-(i), that $\overline{U} \textcircled{\scriptsize T} V$ has an invariant vector $\zeta \in \mathbb{C}^n \otimes H$, and so

$$(\iota \otimes \omega_{\zeta,\zeta})(\overline{U} \textcircled{\scriptsize T} V) = \langle \zeta, \zeta \rangle 1. \tag{5}$$

For $\xi \in H$ and $\alpha \in \mathbb{C}^n$, let $x = \theta_{\alpha,\xi} \in B(H, \mathbb{C}^n)$ be defined by $\theta_{\alpha,\xi}(\eta) = \langle \eta, \xi \rangle \alpha$ for $\eta \in H$. If $y = (\mu \otimes \iota)(U^*(1 \otimes \theta_{\alpha,\xi})V) = 0$ for every $\xi \in H$ and $\alpha \in \mathbb{C}^n$, then for every $\alpha, \beta \in \mathbb{C}^n$ and $\xi, \eta \in H$, we have

$$\begin{aligned} 0 = \langle y\eta, \beta \rangle &= \mu\big((\iota \otimes \omega_{\eta,\beta})(U^*(1 \otimes \theta_{\alpha,\xi})V)\big) = \mu\big((\iota \otimes \omega_{\beta,\alpha})(U)^*(\iota \otimes \omega_{\eta,\xi})(V)\big) \\ &= \mu\big((\iota \otimes \omega_{\alpha,\beta})(\overline{U})(\iota \otimes \omega_{\eta,\xi})(V)\big). \end{aligned}$$

Therefore $(\mu \otimes \iota)(\overline{U} \mathbin{ⓣ} V) = 0$, and so by equation (5)

$$0 = \omega_{\zeta,\zeta}\big((\mu \otimes \iota)(\overline{U} \mathbin{ⓣ} V)\big) = \mu\big((\iota \otimes \omega_{\zeta,\zeta})(\overline{U} \mathbin{ⓣ} V)\big) = \langle \zeta, \zeta \rangle \mu(1) \neq 0.$$

Consequently, $y \neq 0$ for some $\xi \in H$, $\alpha \in \mathbb{C}^n$.

To finish the proof we may argue as in the last part of the proof of [29, Theorem 2.6]. ☐

We now prove a generalisation of conditions (T5) and (T6) in Theorem 5.12.

Theorem 5.15. *Let $\mathbb{G}$ be any locally compact quantum group. Then $\mathbb{G}$ having Property (T) is equivalent to any of the following statements:*

- (T5) *Every finite-dimensional irreducible C*-representation of $C_0^u(\widehat{\mathbb{G}})$ is an isolated point in $\widehat{C_0^u(\widehat{\mathbb{G}})}$.*
- (T6′) *There is a finite-dimensional irreducible C*-representation of $C_0^u(\widehat{\mathbb{G}})$ which is covariant with respect to the scaling automorphism group $(\widehat{\tau}_t^u)$ and is an isolated point in $\widehat{C_0^u(\widehat{\mathbb{G}})}$.*
- (T6) *$C_0^u(\widehat{\mathbb{G}}) \cong B \oplus M_n(\mathbb{C})$ for some C*-algebra B and some $n \in \mathbb{N}_0$ with $\widehat{\tau}_t^u(B) \subset B$ for all $t \in \mathbb{R}$.*

Proof. The proof is based on the same idea as the proof of [10, Theorem 3.6].

(T5) $\Longrightarrow$ (T6′) because covariant irreducible finite-dimensional representations always exist: the counit.

(T6′) $\Longrightarrow$ (T6): Let π be a finite-dimensional irreducible representation of $C_0^u(\widehat{\mathbb{G}})$ which is covariant with respect to the scaling automorphism group. By the equivalence of (i) and (iii) in Proposition 5.1, $C_0^u(\widehat{\mathbb{G}}) \cong \ker \pi \oplus M_n(\mathbb{C})$ for some $n \in \mathbb{N}_0$. Since π is covariant, we have $\widehat{\tau}_t^u(\ker(\pi)) \subset \ker \pi$ for all $t \in \mathbb{R}$, and so (T6) holds.

It remains to prove that (T1) of Theorem 5.12 implies (T5) and that (T6) implies (T2) of Theorem 5.12. The result will then follow from the equivalence of (T1) and (T2) of Theorem 5.12.

(T1) $\Longrightarrow$ (T5): To prove (T5), it is enough to show that (ii) in Proposition 5.1 holds for every irreducible finite-dimensional representation ρ of $C_0^u(\widehat{\mathbb{G}})$. To this end, let π be a representation of $C_0^u(\widehat{\mathbb{G}})$ such that $\rho \prec \pi$. Let $U = (\iota \otimes \rho)(\mathbb{W})$ and $V = (\iota \otimes \pi)(\mathbb{W})$. Since $\mathbb{G}$ has Property (T), it is unimodular by Theorem 6.1, and so $\mathbb{G}$ satisfies the Admissibility Conjecture by Theorem 4.3. Hence U is an irreducible finite-dimensional admissible unitary representation of $\mathbb{G}$ and $U \prec V$. Now by Lemma 5.13 we have that $1 \subset U^c \mathbin{ⓣ} U$ and also $U^c \mathbin{ⓣ} U \prec U^c \mathbin{ⓣ} V$ by Remark 5.4. Since $\mathbb{G}$ has Property (T) and $1 \prec U^c \mathbin{ⓣ} V$, it follows that $1 \subset U^c \mathbin{ⓣ} V$. Thus by Theorem 5.14 there exists a finite-dimensional unitary representation W such that $W \subset U$ and $W \subset V$. Since U is irreducible, this implies that $W = U$, and so $\rho \subset \pi$.

(T6) $\Longrightarrow$ (T2): Suppose that $C_0^u(\widehat{\mathbb{G}}) \cong B \oplus M_n(\mathbb{C})$ and B in invariant under the scaling automorphism group. Let $\mu : C_0^u(\widehat{\mathbb{G}}) \to M_n(\mathbb{C})$ be the irreducible representation corresponding to the summand $M_n(\mathbb{C})$, and let $U = (\iota \otimes \mu)(\mathbb{W})$ be the unitary representation of $\mathbb{G}$

associated with μ. Since $\ker(\mu) \cong B$ and $\widehat{\tau}_t^u(B) \subset B$ for all $t \in \mathbb{R}$, it follows that μ is covariant with respect to the scaling automorphism group of $C_0^u(\widehat{\mathbb{G}})$. Hence U is admissible by Proposition 3.2. The representation π of $C_0^u(\widehat{\mathbb{G}})$ associated to $U^c \circledT U$ is finite-dimensional and covariant with respect to scaling automorphism of $C_0^u(\widehat{\mathbb{G}})$ by Proposition 3.2 since $U^c \circledT U$ is admissible. Therefore, π decomposes into a direct sum of finitely many covariant irreducible representations. Since $1 \subset U^c \circledT U$ by Lemma 5.13, one of the irreducible components is $\widehat{\epsilon}_u$. Therefore $\pi = \bigoplus_{k=0}^m \pi_k$, where $\pi_0 = \widehat{\epsilon}_u$ and π_k is an irreducible finite-dimensional covariant representation for each $k = 1, 2, \dots, m$. By Proposition 5.1, each singleton set $\{\pi_k\}$ is closed in the Fell topology. Towards a contradiction, let us assume that $\widehat{\epsilon}_u$ is not an isolated point. So there is a net

$$(\rho_\alpha)_{\alpha\in\Lambda} \subset \widehat{C_0^u(\widehat{\mathbb{G}})} \setminus \{\widehat{\epsilon}_u, \pi_1, \pi_2, \dots, \pi_m\}$$

such that $(\rho_\alpha)_{\alpha\in\Lambda}$ converges in the Fell topology to $\widehat{\epsilon}_u$. By the definition of closure in the Fell topology, this implies that $\widehat{\epsilon}_u \prec \bigoplus_{\alpha\in\Lambda} \rho_\alpha$, and so

$$\mu = (\mu \otimes \widehat{\epsilon}_u) \circ \chi \circ \widehat{\Delta}_u \prec \bigoplus_{\alpha\in\Lambda} (\mu \otimes \rho_\alpha) \circ \chi \circ \widehat{\Delta}_u,$$

where χ is the flip map. Since μ satisfies condition (iii) in Proposition 5.1 (by definition), it follows by condition (ii) in Proposition 5.1 that

$$\mu \subset \bigoplus_{\alpha\in\Lambda} (\mu \otimes \rho_\alpha) \circ \chi \circ \widehat{\Delta}_u.$$

By irreducibility of μ we have $\mu \subset (\mu \otimes \rho_\alpha) \circ \chi \circ \widehat{\Delta}_u$ for some $\alpha \in \Lambda$. Letting $U_\alpha = (\iota \otimes \rho_\alpha)(\mathbb{W})$, this means that $U \subset U \circledT U_\alpha$. Combining this with Lemma 5.13 it follows that

$$1 \subset U^c \circledT U \subset U^c \circledT U \circledT U_\alpha,$$

so that we have

$$1 \subset U^c \circledT U \circledT U_\alpha = \Big(\bigoplus_{k=0}^m (\iota \otimes \pi_k)(\mathbb{W}) \Big) \circledT U_\alpha.$$

Since $(U^c \circledT U)^c$ is equivalent to $U^c \circledT U$ (recall that U is admissible, Remark 3.5), it follows that

$$1 \subset \Big(\bigoplus_{k=0}^m (\iota \otimes \pi_k)(\mathbb{W}) \Big)^c \circledT U_\alpha.$$

An application of Theorem 5.14 now yields a finite-dimensional unitary representation W such that $W \subset \bigoplus_{k=0}^m (\iota \otimes \pi_k)(\mathbb{W})$ and $W \subset U_\alpha$. Since U_α is irreducible, we have $W = U_\alpha$. On the other hand, π_k is irreducible for $k = 0, 1, 2, \dots, m$, and so $U_\alpha = (\iota \otimes \pi_{k_0})(\mathbb{W})$ for some $k_0 \in \{0, 1, 2, \dots, m\}$. This means that $\rho_\alpha = \pi_{k_0}$ which is a contradiction. Thus $\widehat{\epsilon}_u$ must be an isolated point. □

6. Properties of quantum groups with Property (T)

In this section we prove several properties shared by quantum groups with Property (T). We include a very short proof of the fact that a quantum group with Property (T) is unimodular (see [9, Section 6] and [22, Proposition 7]). We consider unimodular locally compact quantum groups as well as quantum groups arising through the bicrossed product construction as discussed in Subsection 4.1 and prove a variation of Theorem 5.15 and improved versions of the quantum versions of Bekka–Valette theorem [9, Theorem 4.8] (characterising Property

(T) in terms of non-existence of almost invariant vectors for weakly mixing representations) and Kerr–Pichot theorem [9, Theorem 4.9] (characterising Property (T) in terms of density properties of weakly mixing representations) for these quantum groups.

6.1. **Quantum groups with Property (T) are unimodular.** It is a well-known fact that a locally compact group G with Property (T) is unimodular [8, Corollary 1.3.6-(ii)]. The proof of this result makes use of the fact that if G has Property (T) and admits a continuous homomorphism into a locally compact group H with dense range, then H has Property (T) [8, Theorem 1.3.4]. A version of [8, Theorem 1.3.4] for locally compact quantum groups has been obtained in [18, Theorem 5.7]. Using this, it seems plausible that one can prove that Property (T) for quantum groups implies unimodularity similarly to the classical case. However, the proofs in the case of quantum groups have proceeded differently.

To the best of our knowledge, the first result in this direction for quantum groups was proven for discrete quantum groups [22, Proposition 7]. This was subsequently generalised to second countable locally compact quantum groups [9, Section 6]. The proof in the case of a locally compact quantum group, as given in [9], proceeds via showing that non-unimodular locally compact quantum groups always admit a weakly mixing representation that weakly contains the trivial representation, as a consequence the quantum group cannot have Property (T).

We give a very short proof of the fact that Property (T) implies unimodularity, using a completely different technique, which does not require the second countability assumption.

Theorem 6.1. *Let $\mathbb{G}$ be a locally compact quantum group with Property (T). Then $\mathbb{G}$ is unimodular.*

Proof. Let δ denote the modular element of $\mathbb{G}$, so that δ is a strictly positive element affiliated to $C_0(\mathbb{G})$ [27, Definition 7.11]. By [27, Proposition 7.12], for all $s, t \in \mathbb{R}$,

(i) $\Delta(\delta^{is}) = \delta^{is} \otimes \delta^{is}$,
(ii) $\tau_t(\delta^{is}) = \delta^{is}$.

Note that $\delta^0 = 1$ is the trivial representation of $\mathbb{G}$. Relation (i) implies that for all $s \in \mathbb{R}$, δ^{is} is a 1-dimensional unitary representation of $\mathbb{G}$, so there exists C*-representations $\pi_s : C_0^u(\widehat{\mathbb{G}}) \to \mathbb{C}$ such that $(\iota \otimes \pi_s)(\mathbb{W}) = \delta^{is}$. For all $\omega \in B(L^2(\mathbb{G}))_*$, it follows that

$$\lim_{s\to 0} \pi_s((\omega \otimes \iota)(\mathbb{W})) = \lim_{s\to 0} \omega(\delta^{is}) = \omega(1) = \widehat{\epsilon}_u((\omega \otimes \iota)(\mathbb{W})),$$

where $\widehat{\epsilon}_u$ is the counit of $C_0^u(\widehat{\mathbb{G}})$. Since the family $\{\pi_s\}_{s\in\mathbb{R}}$ is uniformly bounded and the elements of the form $(\omega \otimes \iota)(\mathbb{W})$ for $\omega \in B(L^2(\mathbb{G}))_*$ are norm-dense in $C_0^u(\widehat{\mathbb{G}})$ [26, Equation (5.2)], it follows that $\lim_{s\to 0} \pi_s(x) = \widehat{\epsilon}_u(x)$ for all $x \in C_0^u(\widehat{\mathbb{G}})$.

By [26, Proposition 9.1], $(\tau_t \otimes \iota)(\mathbb{W}) = (\iota \otimes \widehat{\tau}^u_{-t})\mathbb{W}$ for all $t \in \mathbb{R}$, where $(\widehat{\tau}^u_t)_{t\in\mathbb{R}}$ is the universal scaling group of $\widehat{\mathbb{G}}$. It then follows by relation (ii) above that $\pi_t(\widehat{\tau}^u_s(x)) = \pi_t(x)$ for all $s, t \in \mathbb{R}$ and $x \in C_0^u(\widehat{\mathbb{G}})$, so that for each $t \in \mathbb{R}$, $\pi_t : C_0^u(\widehat{\mathbb{G}}) \to \mathbb{C}$ is covariant in the sense of Remark 3.4. Since $\mathbb{G}$ has Property (T) by hypothesis, $\widehat{\epsilon}_u$ is isolated in $\widehat{C_0^u(\widehat{\mathbb{G}})}$. Since $\lim_{t\to 0} \pi_t(x) = \widehat{\epsilon}_u(x)$ for all $x \in C_0^u(\widehat{\mathbb{G}})$, it follows that the net $(\pi_t)_{t\in\mathbb{R}}$ converges to $\widehat{\epsilon}_u$ in the Fell topology of $\widehat{C_0^u(\widehat{\mathbb{G}})}$. As $\widehat{\epsilon}_u$ is isolated, we must have $\pi_t = \widehat{\epsilon}_u$ for all $t \in \mathbb{R}$ with $|t|$ sufficiently small, that is, $\delta^{it} = 1$ for all $t \in \mathbb{R}$ with $|t|$ sufficiently small. It follows that in fact $\delta^{it} = 1$ for all $t \in \mathbb{R}$, and so that $\delta = 1$, as required. $\square$

6.2. **Other aspects of quantum groups with Property (T).** Combining Proposition 3.2, Theorem 4.3 and Theorem 5.15 we have the following.

Corollary 6.2. *Let $\mathbb{G}$ be a locally compact quantum group that is either unimodular or arises through the bicrossed product construction as described in Theorem 4.2. Then $\mathbb{G}$ having Property (T) is equivalent to either of the following statements:*

(T6′) *There is a finite-dimensional irreducible C^*-representation of $C_0^u(\widehat{\mathbb{G}})$ that is an isolated point in $\widehat{C_0^u(\widehat{\mathbb{G}})}$.*

(T6) *$C_0^u(\widehat{\mathbb{G}}) \cong B \oplus M_n(\mathbb{C})$ for some C^*-algebra B and some $n \in \mathbb{N}_0$.*

A unitary representation of a locally compact quantum group is *weakly mixing* if it admits no nonzero admissible finite-dimensional subrepresentation (see [43]). Corollary 6.2 together with [9, Lemma 3.6] gives the Bekka–Valette theorem for all unimodular locally compact quantum groups. This was known before only for quantum groups with trivial scaling automorphism group [9, Theorem 4.8].

Corollary 6.3. *Let $\mathbb{G}$ be a second countable locally compact quantum group that is either unimodular or arises through the bicrossed product construction as described in Theorem 4.2. Then $\mathbb{G}$ has Property (T) if and only if every weakly mixing unitary representation of $\mathbb{G}$ fails to have almost invariant vectors.*

Combining Corollary 6.3 with [9, Theorems 3.7 & 3.8] we have the Kerr–Pichot theorem for unimodular quantum groups with non-trivial scaling group. Also this was known before only for the case of quantum groups with trivial scaling group [9, Theorem 4.8].

Given a second countable locally compact quantum group $\mathbb{G}$ and a Hilbert space H, denote the set of all unitary representations of $\mathbb{G}$ on H by $\mathrm{Rep}(\mathbb{G}, H)$ and the set of all weakly mixing unitary representations of $\mathbb{G}$ on H by $W(\mathbb{G}, H)$.

Corollary 6.4. *Let $\mathbb{G}$ be a second countable locally compact quantum group that is either unimodular or arises through the bicrossed product construction as described in Theorem 4.2. Let H be a separable infinite-dimensional Hilbert space. If $\mathbb{G}$ does not have Property (T), then $W(\mathbb{G}, H)$ is a dense G_δ-set in $\mathrm{Rep}(\mathbb{G}, H)$. If $\mathbb{G}$ has Property (T), then $W(\mathbb{G}, H)$ is closed and nowhere dense in $\mathrm{Rep}(\mathbb{G}, H)$.*

References

[1] Y. Arano, Unitary spherical representations of Drinfeld doubles, to appear in *J. Reine. Angew. Math.*

[2] Y. Arano, Comparison of unitary duals of Drinfeld doubles and complex semisimple Lie groups, *Comm. Math. Phys.* **351** (2017), 1137–1147.

[3] S. Baaj and G. Skandalis, Unitaires multiplicatifs et dualité pour les produits croisés de C*-algèbres, *Ann. Sci. École Norm. Sup.* **26** (1993), 425–488.

[4] S. Baaj and S. Vaes, Double crossed products of locally compact quantum groups, *J. Inst. Math. Jussieu* **4** (2005), 135–173.

[5] E. Bédos, R. Conti and L. Tuset, On amenability and co-amenability of algebraic quantum groups and their corepresentations, *Canad. J. Math.* **57** (2005), 17–60.

[6] E. Bédos and L. Tuset, Amenability and coamenability for locally compact quantum groups, *Int. J. Math* **14** no. 8 (2003), 865–884.

[7] B. Bekka and A. Valette, Kazhdan's property (T) and amenable representations, *Math. Z.* **212** (1993), 293–299.

[8] B. Bekka, P. de la Harpe and A. Valette, *Kazhdan's Property (T)*, Cambridge University Press, Cambridge, 2008.

[9] M. Brannan and D. Kerr, Quantum groups, Property (T), and weak mixing, *Comm. Math. Phys.*, to appear, see `https://doi.org/10.1007/s00220-017-3037-0`
[10] X. Chen and C. K. Ng, Property T for general locally compact quantum groups, *Internat. J. Math.* **26**, 1550024 (2015).
[11] A. Connes and B. Weiss, Property T and asymptotically invariant sequences, *Israel J. Math.* **37** (1980), 209–210.
[12] C. D'Antoni and L. Zsidó, Groups of linear isometries on multiplier C*-algebras, *Pacific J. Math.* **193** (2000), 279–307.
[13] A. V. Daele, Discrete quantum groups, *J. Algebra* **180**, (1996) 431–444.
[14] B. Das and M. Daws, Quantum Eberlein compactifications and invariant means, *Indiana Univ. Math. J.* **65** (2016), 307–352.
[15] M. Daws, Remarks on the quantum Bohr compactification, *Illinois J. Math.* **57** (2013), 1131–1171.
[16] M. Daws, P. Fima, A. Skalski and S. White, The Haagerup property for locally compact quantum groups, *J. Reine. Angew. Math.* **711** (2016), 189–229.
[17] M. Daws and P. Salmi, Completely positive definite functions and Bochner's theorem for locally compact quantum groups, *J. Funct. Anal.* **264** (2013), 1525–1546.
[18] M. Daws, A. Skalski and A.Viselter, Around property (T) for quantum groups, *Comm. Math. Phys.* **353** (2017), no. 1, 69–118.
[19] M. Enock and J. M. Schwartz, *Kac algebras and duality of locally compact groups*, Springer-Verlag, Berlin, 1992.
[20] P. Eymard, L'algèbre de Fourier d'un groupe localement compact, *Bull. Soc. Math. France* **92** (1964), 181–236.
[21] J. M. G. Fell, The dual spaces of C*-algebras, *Trans. Amer. Math. Soc.* **94** (1960), 365–403.
[22] P. Fima, Kazhdan's Property T for Discrete Quantum Groups, *Internat. J. Math.* **21** (2010), 47–65.
[23] E. Glasner and B. Weiss, Kazhdan's property (T) and the geometry of the collection of invarinat measures, *Geom. Funct. Anal.* **7** (1997), 917–935.
[24] D. Kerr and M. Pichot, Asymptotic abelianness, weak mixing, and property T, *J. Reine. Angew. Math.* **623** (2008), 213–235.
[25] J. Kustermans, Locally compact quantum groups, *Quantum Independent Increment Processes I*, Eds. M. Schürmann and U. Franz, Springer-Verlag, Berlin, 2005, pp. 99–180.
[26] J. Kustermans, Locally compact quantum groups in the universal setting, *Internat. J. Math.* **12** (2001), 289–338.
[27] J. Kustermans and S. Vaes, Locally compact quantum groups, *Ann. Sci. École Norm. Sup.* **33** (2000), 837–934.
[28] J. Kustermans and S. Vaes, Locally compact quantum groups in the von Neumann algebraic setting, *Math. Scand.* **92** (2003), 68–92.
[29] D. Kyed and P. M. Sołtan, Property (T) and exotic quantum group norms, *J. Noncommut. Geom.* **6** (2012), 773–800.
[30] A. Maes and A. Van Daele, Notes on compact quantum groups, *Nieuw Arch. Wisk.* **16** (1998), 73–112.
[31] T. Masuda, Y. Nakagami and S. L. Woronowicz, A C*-algebraic framework for quantum groups, *Internat. J. Math.* **14** (2003), 903–1001.
[32] R. Meyer, S. Roy, S. L. Woronowicz, Homomorphisms of quantum groups, *Munster J. Math.* **5** (2012), 1–24.
[33] S. Neshveyev and L. Tuset, *Compact quantum groups and their representation categories*, Société Mathématique de France, Paris, 2013.
[34] S. Neshveyev and M. Yamashita, Drinfeld center and representation theory for moniodal categories, *Comm. Math. Phys.* **345** (2016), 385–434.
[35] S. Petrescu and M. Joiţa, Property (T) for Kac algebras, *Rev. Roumaine Math. Pures. Appl.* **37** (1992), 163–178.
[36] G. Pisier, *Similarity problems and completely bounded maps. Second, expanded edition. Includes the solution to "The Halmos problem"*, Springer-Verlag, Berlin, 2001.
[37] S. Popa and S. Vaes, Representation theory for subfactors, λ-lattices and C*-tensor categories, *Comm. Math. Phys.* **340** (2015), 1239–1280.
[38] W. Rudin, *Functional Analysis*, Second edition, McGraw-Hill, Inc., New York, 1991.
[39] P. Sołtan, Quantum Bohr compactification, *Illinois J. Math.* **49** (2005), 1245–1270.

[40] P. Sołtan and S.L. Woronowicz, From multiplicative unitaries to quantum groups II, *J. Funct. Anal.* **252** (2007), 42–67.
[41] T. Timmermann, *An Invitation to Quantum Groups and Duality*, EMS Textbooks in Mathematics, (2008).
[42] S. Vaes and L. Vainerman, Extensions of locally compact quantum groups and the bicrossed product construction, *Adv. Math.* **175** (2003), 1–101.
[43] A. Viselter, Weak mixing for locally compact quantum groups. *Ergodic Theory Dynam. Systems* **37** (2017), 657–1680.
[44] P. S. Wang, On isolated points in the dual spaces of locally compact groups, *Math. Ann.* **218** (1975), 19–34.
[45] S. Wang, Free products of compact quantum groups, *Comm. Math. Phys.* **167** (1995), 671–692.
[46] S. L. Woronowicz, From multiplicative unitaries to quantum groups, *Internat. J. Math.* **7** (1996), 127–149.
[47] S. L. Woronowicz, Compact quantum groups, *Symétries quantiques (Les Houches, 1995)*, 845–884 North-Holland, Amsterdam, 1998.
[48] S. L. Woronowicz, Twisted SU(2) group, An example of a noncommutative differential calculus, *Publ. Res. Inst. Math. Sci.* **23** (1987), 117–181.
[49] S. L. Woronowicz, A remark on compact matrix quantum groups, *Lett. Math. Phys.* **21** (1991), 35–39.
[50] R. J. Zimmer, *Ergodic Theory and Semisimple Groups*, Monographs in Mathematics, **81** (1984), Birkhäuser Verlag, Basel.

Department of Mathematical Sciences, University of Oulu, Finland.

Instytut Matematyczny, Uniwersytet Wrocławski, Poland.
E-mail address: biswarup.das@math.uni.wroc.pl

Jeremiah Horrocks Institute, University of Central Lancashire, UK.
E-mail address: matt.daws@cantab.net

Deprtment of Mathematical Sciences, University of Oulu, Finland.
E-mail address: pekka.salmi@oulu.fi