

# The Boundary Element Method in Excel for Teaching Vector Calculus and Simulation

Stephen Kirkup

**Abstract**—This paper discusses the implementation of the boundary element method (BEM) on an Excel spreadsheet and how it can be used in teaching vector calculus and simulation. There are two separate spreadsheets, within which Laplace equation is solved by the BEM in two dimensions (LIBEM2) and axisymmetric three dimensions (LBEMA). The main algorithms are implemented in the associated programming language within Excel, Visual Basic for Applications (VBA). The BEM only requires a boundary mesh and hence it is a relatively accessible method. The BEM in the open spreadsheet environment is demonstrated as being useful as an aid to teaching and learning. The application of the BEM implemented on a spreadsheet for educational purposes in introductory vector calculus and simulation is explored. The development of assignment work is discussed, and sample results from student work are given. The spreadsheets were found to be useful tools in developing the students' understanding of vector calculus and in simulating heat conduction.

**Keywords**—Boundary element method, Laplace equation, vector calculus, simulation, education

## I. INTRODUCTION

FOR some time, the spreadsheet has been utilised by scientists and engineers as a powerful environment for exploring numerically-based topics. There are a number of spreadsheet packages available, but the most prevalent one at the time of writing is Microsoft Excel. The interactive nature of the spreadsheet, and the immediacy of data and graphs, is the basis of its usefulness as a tool for exploring a topic in science and engineering, and hence its educational predisposition. There are a number of texts on the application of Excel to scientific and engineering problems available [1]-[3]. These texts are a learning resource in which the authors' clearly regard the spreadsheet environment as conducive to education. A recent paper by Niazkar and Afzali [4] reviews various recent application areas for spreadsheets. The usefulness of spreadsheet applications in enhancing engineering education has been the subject of research at the University of Central Lancashire for some time [5], [6].

The spreadsheet is useful for evaluating formulae, a facility that is required in mathematics, science and engineering. However, when mathematical models include, for example, integrals, differential equations or integral equations then, in general, numerical methods are required. The spreadsheet environment has also been explored for implementing numerical methods, including educational purposes [3], [7]-[9]. A wide-ranging set of problems in science and engineering is modelled by partial differential equations (PDEs). PDEs

generally provide a physical model within a prescribed spatial domain in one, two or three dimensions, and often also in time. The spreadsheet naturally lends itself to the lower dimensional problems, given the two-dimensional nature of the computer screen and spreadsheet environment. The spreadsheet is therefore most applicable for the educational side of PDEs, and perhaps restrictive for the three-dimensional domains that will be encountered in practice.

In this work, the BEM [10] is implemented in Excel. The BEM is a method for solving PDEs. The BEM is not as widely-applicable as the more well-known methods for solving PDEs - the finite element method and the finite difference method. However, the BEM has the special property that only the boundary requires discretisation, and hence, the BEM is often viewed as potentially more efficient than the alternative methods, as less elements or nodes are required. More importantly, particularly from the user or educational viewpoint of this work, the requirement to only mesh the boundary also means that the BEM is a significantly more accessible method.

There has been little development of the BEM in a spreadsheet environment for some time. The main work in this area is that of Davies and Crann in the 1990s [11]. The work of Davies and Crann was focused on using the Excel spreadsheet to teach the BEM itself. Whilst the work in this paper may be extended to that aim, the focus on the educational purpose of the BEM implemented on a spreadsheet is that its accessibility facilitates teaching and learning in elementary vector calculus and simulation. The motivation for the work of Davies and Crann was that of using the spreadsheet to direct students of the BEM to focus on the implementation stages of the method itself, rather than on the development of computer code. In this paper, the focus is not just on hiding the complexity of coding the BEM, but also hiding the BEM itself so that students are not distracted from learning about vector calculus and elementary simulation.

Spreadsheets have been available for several decades. Davies and Crann state that their implementation of the BEM could be transferred from Excel to other spreadsheet products, emphasising the portability of their work. However, spreadsheets evolve, with new versions every few years, and there is a concern about the maintenance of applications. In this work, the main computations are carried out using the programming language VBA, that accompanies the Excel spreadsheet and can be activated from the spreadsheet. The sheets and cells of the accompanying spreadsheets are only used to set the input, communicate the output, and hold intermediate data. The

outcome of this approach is that greater generality can be established, and the codes are as maintainable as any computer codes. The Excel spreadsheets in this work are more maintainable than the works of Davis and Crann but are not transferable between various spreadsheet platforms.

The simplest equation that can be solved by the BEM is the Laplace equation. The spreadsheet of Davis and Crann solves Laplace equation in a two-dimensional interior domain. In this work, a spreadsheet LIBEM2.xlsm that enables the solution of Laplace equation in the two-dimensional interior domain and a spreadsheet LBEMA.xlsm that solves Laplace equation in axisymmetric three-dimensions are introduced. LIBEM2 and LBEMA are available and free to download from the author's website [8] as open-source. LIBEM2 was used as a practical tool in a lecture course given by the author at the LD College of Engineering in Ahmedabad [12] and LIBEM2 and LBEMA have recently been used in assignments for year two and year three School of Engineering (SC2153 and SC3007) students at the University of Central Lancashire. This work extends the author's previous work on development and communication in the BEM [10], [13]-[16].

In this paper, the boundary integral equations for the solution of the interior two-dimensional Laplace equation are outlined in Section II. In Section III, it is shown that these equations can be discretised in order to develop the equations that are the basis of the two-dimensional BEM. LIBEM2, the spreadsheet for solving two-dimensional interior Laplace problems, is introduced in Section IV. In Section V, the work is extended to axisymmetric three-dimensional problems, and the LBEMA spreadsheet is introduced. The educational application of the spreadsheets in the area of vector calculus and simulation is considered in Section VI with the results from the students' assignment work.

## II. THE BOUNDARY INTEGRAL EQUATION FORMULATION FOR 2D INTERIOR PROBLEMS

In this section, the boundary element solution of Laplace equation:

$$\nabla^2 \varphi = 0, \quad (1)$$

for two-dimensional problems, and axisymmetric three-dimensional problems is outlined. Traditionally, there have been two variations of the BEM, termed the *Direct Method* and the *Indirect Method*. Since there is a vast overlap in the computation in these methods, and for the long-term various uses of the spreadsheets, both methods are implemented. LIBEM2 solves the interior Laplace problem in two-dimensional space and is effectively the 'entry level' for learners.

### A. Integral Equation Formulations of the 2D Interior Laplace Problem

For the interior problem, Laplace equation (1) governs the interior domain  $D$  enclosed by a boundary  $S$ , as shown in Fig. 1. The solution must also satisfy a boundary condition, and it is important in terms of maintaining the generality of the method

that this is in a general (Robin) form:

$$\alpha(\mathbf{p})\varphi(\mathbf{p}) + \beta(\mathbf{p})\frac{\partial\varphi}{\partial n_p}(\mathbf{p}) = f(\mathbf{p}) \quad (\mathbf{p} \in S). \quad (2)$$

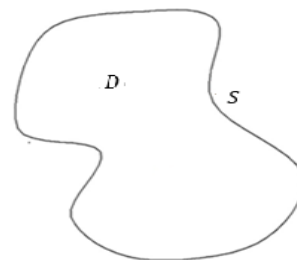


Fig. 1 Illustration of the interior domain [10]

In the direct BEM, Laplace equation is replaced by an equivalent integral equation of the form:

$$\int_S \frac{\partial G(\mathbf{p}, \mathbf{q})}{\partial n_q} \varphi(\mathbf{q}) dS_q + \frac{1}{2} \varphi(\mathbf{p}) = \int_S G(\mathbf{p}, \mathbf{q}) \frac{\partial \varphi(\mathbf{q})}{\partial n_q} dS_q \quad (\mathbf{p} \in S). \quad (3a)$$

$$\int_S \frac{\partial G(\mathbf{p}, \mathbf{q})}{\partial n_q} \varphi(\mathbf{q}) dS_q + \varphi(\mathbf{p}) = \int_S G(\mathbf{p}, \mathbf{q}) \frac{\partial \varphi(\mathbf{q})}{\partial n_q} dS_q \quad (\mathbf{p} \in D). \quad (3b)$$

The terminology  $\frac{\partial^*}{\partial n_q}$  represents the partial derivative of the function  $*$  with respect to the unit outward normal at the point  $\mathbf{q}$  on the boundary. The function  $G$  is known as a Green's function. Physically,  $G(\mathbf{p}, \mathbf{q})$  represents the effect observed at a point  $\mathbf{p}$  of a unit source at the point  $\mathbf{q}$ . For the Laplace equation, the Green's function is denoted by  $G$  and is defined as  $G(\mathbf{p}, \mathbf{q}) = -\frac{1}{2\pi} \ln r$  for two-dimensional Laplace problems, where  $r = |\mathbf{q} - \mathbf{p}|$ .

The equivalent indirect formulation is obtained by writing  $\varphi(\mathbf{p})$  as a single layer potential:

$$\varphi(\mathbf{p}) = \int_S G(\mathbf{p}, \mathbf{q}) \sigma(\mathbf{q}) dS_q \quad (\mathbf{p} \in SUD), \quad (4a)$$

where the boundary function  $\sigma$  is the single layer potential with no particular physical representation. By differentiating with respect to the outward normal to the boundary for a point  $\mathbf{p} \in D$  and taking the limit, as the point approaches the boundary along a normal, returns the following equation:

$$\frac{\partial \varphi(\mathbf{p})}{\partial n_p} = \int_S \frac{\partial G(\mathbf{p}, \mathbf{q})}{\partial n_q} \sigma(\mathbf{q}) dS_q + \frac{1}{2} \sigma(\mathbf{p}) \quad (\mathbf{p} \in S). \quad (4b)$$

### B. The Integral Equation Formulations in Operator Notation

Integral operators provide a useful shorthand notation for writing integral equations and form a useful basis for generalisation in developing BEM software. Applying the integral operator to a function  $\zeta$ , defined on a boundary  $\Gamma$ ,

$$\int_{\Gamma} G(\mathbf{p}, \mathbf{q}) \zeta(\mathbf{q}) dS_q = \mu(\mathbf{p}),$$

returning a function  $\mu$ . This may be written in a simplified form,

$$\{L\zeta\}_\Gamma(\mathbf{p}) = \mu(\mathbf{p}), \quad (5)$$

where  $L$  represents the integral operator and the subscript ( $\Gamma$ ) refers to the domain of integration.  $\Gamma$  is used as a variable, representing either a whole boundary or a part of the boundary. The other Laplace integral operators required in this work are defined as follows:

$$\{M\zeta\}_\Gamma(\mathbf{p}) = \int_\Gamma \frac{\partial G(\mathbf{p}, \mathbf{q})}{\partial n_q} \zeta(\mathbf{q}) dS_q, \quad (6)$$

$$\{M^t \zeta\}_\Gamma(\mathbf{p}; \mathbf{v}_p) = \frac{\partial}{\partial v_p} \int_\Gamma G(\mathbf{p}, \mathbf{q}) \zeta(\mathbf{q}) dS_q, \quad (7)$$

where  $\mathbf{v}_p$  is any unit vector.

In operator notation, the direct integral equation formulation (3a) and (3b) can be written in the following form:

$$\left\{ \left( M + \frac{1}{2} I \right) \varphi \right\}_S(\mathbf{p}) = \{L v\}_S(\mathbf{p}) \quad (\mathbf{p} \in S), \quad (8a)$$

$$\varphi(\mathbf{p}) = \{L v\}_S - \{M \varphi\}_S \quad (\mathbf{p} \in D), \quad (8b)$$

where  $v(\mathbf{q}) = \frac{\partial \varphi(\mathbf{q})}{\partial n_q}$ . Similarly, for the indirect formulation (4a) and (4b),

$$\varphi(\mathbf{p}) = \{L \sigma\}_S \quad (\mathbf{p} \in S \cup D), \quad (9a)$$

$$v(\mathbf{p}) = \left\{ \left( M^t + \frac{1}{2} I \right) \sigma \right\}_S \quad (\mathbf{p} \in S). \quad (9b)$$

### III. THE BEM FOR 2D PROBLEMS

The boundary  $S$  is assumed to be expressed and approximated by a set of panels:

$$S \approx \tilde{S} = \sum_{j=1}^n \Delta \tilde{S}_j. \quad (10)$$

Usually the panels have a characteristic form and cannot represent a given boundary exactly. The simplest method of achieving this is through each  $\Delta \tilde{S}_j$  being a straight line, and this is the method that is used in LIBEM2. Fig. 2 illustrates this method of approximation on the boundary in Fig. 1.



Fig. 2 The boundary represented by a set of straight-line panels [10]

The functions defined on the boundary, that occur in the boundary integral equation formulations, are also represented or approximated by a simple functional form on each panel, for example in the method of collocation. In this work, the

boundary functions are represented by a constant on each panel, with the collocation point at the centre. The *element* is defined by the form of the panel and the representation the boundary functions.

#### A. Direct and Indirect BEMs

As stated earlier, the first stage of the BEM involves finding further information on the boundary  $S$ . For the direct BEM solution of the interior Laplace problem, that is developed in this section, the initial stage involves solving the boundary integral equation (3a), returning (approximations to) both  $\varphi$  and  $\frac{\partial \varphi}{\partial n}$  on  $S$ . The second stage of the BEM involves finding the solution at any chosen points in the domain. The substitution of representations for the boundary functions in the integral equation reduces it to discrete form.

The simplifications allow us to re-write (8a) as the approximation

$$\sum_{j=1}^n \left\{ \left( M + \frac{1}{2} I \right) e \right\}_{\Delta \tilde{S}_j}(\mathbf{p}) \varphi_j \approx \sum_{j=1}^n \{L e\}_{\Delta \tilde{S}_j}(\mathbf{p}) v_j \quad (\mathbf{p} \in \tilde{S})$$

where  $e$  is the unit function ( $e \equiv 1$ ).

The boundary function is approximated or represented by a constant located at the central point of each panel (the collocation point). Computing the discrete forms of the relevant integral operators, with  $\mathbf{p}$  taking the value of all the collocation points, results in the following system

$$\sum_{j=1}^n \left\{ \left( M + \frac{1}{2} I \right) e \right\}_{\Delta \tilde{S}_j}(\mathbf{p}_{Si}) \varphi_j \approx \sum_{j=1}^n \{L e\}_{\Delta \tilde{S}_j}(\mathbf{p}_{Si}) v_j \quad (\mathbf{p}_{Si} \in \tilde{S})$$

for  $i = 1, 2, \dots, n$  is obtained, by putting  $\mathbf{p} = \mathbf{p}_{Si}$  in the previous approximation.

This system can now be written in the matrix-vector form,

$$\left( M_{SS} + \frac{1}{2} I \right) \hat{\varphi}_S = L_{SS} \hat{v}_S \quad (11)$$

with the matrix components defined by  $[L_{SS}]_{ij} = \{L e\}_{\Delta \tilde{S}_j}(\mathbf{p}_{Si})$ ,  $[M_{SS}]_{ij} = \{M e\}_{\Delta \tilde{S}_j}(\mathbf{p}_{Si})$ . The vectors  $\hat{\varphi}_S$  and  $\hat{v}_S$  are representative or approximate values of  $\varphi$  and  $v$  at the collocation points. In the first stage of the BEM, the system (11) is solved alongside the discrete form of the boundary condition (2):

$$\alpha_i \varphi_i + \beta_i v_i = f_i \quad \text{for } i = 1, 2, \dots, n. \quad (12)$$

The matrix components in (11) are definite integrals that usually need to be computed by numerical integration. On solution of (11) and (12), the approximation to the boundary data is known at the collocation points.

Once the approximations to the functions on the boundary are known, after completing the initial stage of the direct BEM, the domain solution can be found. In the case of the interior Laplace problem, equation (3b/8b) will yield the domain solution. Similarly, the discrete equivalent of (8b) may be derived:

$$\underline{\hat{\varphi}}_P = L_{PS} \underline{\hat{v}}_S - M_{PS} \underline{\hat{\varphi}}_S \quad (13)$$

where  $[L_{PS}]_{ij} = \{Le\}_{\Delta S_j}(\mathbf{p}_{Di})$ ,  $[M_{PS}]_{ij} = \{Me\}_{\Delta S_j}(\mathbf{p}_{Di})$  and the  $\mathbf{p}_{Di}$  are the points in the domain  $D$ , where the solution is sought.

Similarly, for the indirect method, the discrete forms of (9a) and (9b), for obtaining the approximation to the boundary functions, are as follows

$$\underline{\hat{\varphi}}_S = L_{SS} \underline{\hat{\varphi}}_S, \quad (14a)$$

$$\underline{\hat{v}}_S = \left( M_{SS}^t + \frac{1}{2}I \right) \underline{\hat{\varphi}}_S, \quad (14b)$$

where the  $M_{SS}^t$  matrix is the discrete equivalent of the  $M^t$  operator, defined in a similar way as the matrices  $L_{SS}$  and  $M_{SS}$  for their respective operators. Equations (14) are solved simultaneously with the discrete boundary condition (12) in order to compute the approximation to the layer potential  $\underline{\hat{\varphi}}_S$ . The solution at the domain points can then be found using the discrete equivalent of (9a):

$$\underline{\hat{\varphi}}_P = L_{PS} \underline{\hat{\varphi}}_S. \quad (14c)$$

#### IV. LIBEM2: THE SOLUTION OF THE 2D LAPLACE EQUATION IN EXCEL

In this section, the Excel spreadsheet LIBEM2 is introduced. This spreadsheet solves the two-dimensional interior Laplace problem for a domain of any shape and with a generalised boundary condition. The Laplace problem is input from the first sheet *Set Problem*; the boundary, boundary condition and the interior points (at which the solution is sought) are all input from this sheet. The sheet *Sketch* illustrates the boundary and the interior points and hence is most useful for visually checking for geometrical errors. The spreadsheet allows for internal boundaries, as we will see with the test problems, and the sheet *Closed Boundaries* separates these, and this sheet is also useful for checking the geometry. The computed solutions are returned to the sheets *Direct Solution* and *Indirect Solution*. The focus of this paper is on using the spreadsheet to teach vector calculus and simulation and, for these purposes, the student does not need to go beyond these areas of the spreadsheet.

The computations within the BEM are activated from buttons on the *Set Problem* sheet. These computations are carried out using the VBA programming language that accompanies Excel. In the previous section, it was shown that the BEM involves evaluating matrices and solving the ensuing systems of equations. The data corresponding to these are placed on identified sheets of the spreadsheet. These are useful for students learning the BEM, but they are not required for the purposes of this paper.

The matrices that were introduced in the previous section, that are computed when the method is executed, are listed on the individual sheets, each labelled with the corresponding identifier. The solution of the equations corresponding to the

direct problem requires a column-exchanging method. Both methods require the LU factorisation of the resulting system and the data recording this is also stored on various sheets. If  $f(\mathbf{p})$  is changed in the boundary condition, then these values may be placed on the *New Condition* sheet, and the results from this may be computed in a fraction of the time that is required to execute a new problem from the *Set Problem* sheet. However, none of the information in this paragraph is required for the purposes of teaching and learning vector calculus and simulation.

##### A. Test Problem

In order to introduce the spreadsheet, a simple test problem is placed on the *Set Problem* sheet. The problem is illustrated in Fig. 3, in which the boundary is a unit square, and the Dirichlet boundary conditions  $\varphi=10$  and  $\varphi=20$  are placed on the left and right sides of the square and the Neumann boundary condition of  $\partial\varphi/\partial n=0$  is set on the upper and lower sides. The interior points, at which the solution is sought, are also shown in Fig. 3, these are the points (0.25, 0.25), (0.75, 0.25), (0.25, 0.75), (0.75, 0.75) and (0.5, 0.5).

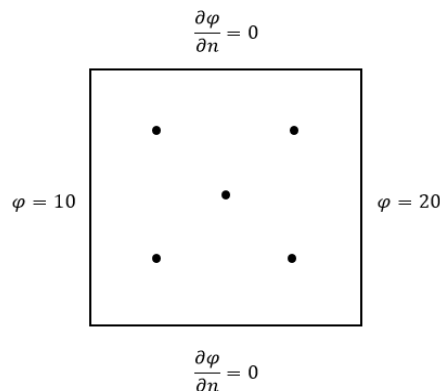


Fig. 3 The 2D Test Problem

From the mathematical point of view, the solution is  $\varphi = 10 + x$ . This can easily be shown to be a solution of Laplace equation and satisfying the left and right boundary condition. Given the identity  $\frac{\partial\varphi}{\partial n} = \nabla\varphi \cdot \mathbf{n}$ , it can be shown that the Neumann conditions are also satisfied as  $\nabla\varphi = \begin{pmatrix} 10 \\ 0 \end{pmatrix}$  and  $\mathbf{n} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$  for the upper surface and  $\mathbf{n} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$  for the lower surface. The solutions at the interior points are  $\varphi = 12.5, 15, \text{ and } 17.5$ .

In order to motivate the practical application of the spreadsheet, the test problem may also be considered in a physical or engineering sense, and it is useful to relate the physical with the mathematical solution. For example, it may be interpreted as a steady-state heat conduction problem, with  $\varphi$  as the temperature. The square may be thought of as a metal plate (insulated on the planar surfaces or a square prism), with the set temperature of 10 °C on the left side and 20 °C on the right side. The  $\partial\varphi/\partial n=0$  on the upper and lower sides may be interpreted as no heat flow or insulation. The solution is just as anyone would expect, with  $\varphi = 12.5$  °C, 15 °C, or 17.5 °C at the selected interior points.

*B. Setting up the Test Problem on the Spreadsheet*

The boundary is defined on the spreadsheet by a set of nodes and panels that are defined in the Nodes and Panels columns of the spreadsheet. The method for describing the boundary is illustrated in Fig. 4. The outer boundary must be defined in the clockwise direction (if there are any inner boundaries then they must be defined in the counter-clockwise direction). For the test problem above, with the boundary of the unit square, the nodes on the boundary are enumerated 1, 2, 3, 4, with the first node having coordinates (0,0.000).

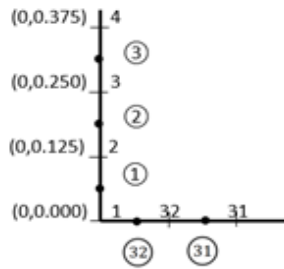


Fig. 4 The nodes and panels that describe the boundary

If the square is defined by 32 panels of equal size then the node 2 has coordinates (0,0.125), as shown in Fig. 4. The coordinates of nodes 3 and 4 are also illustrated. 32 nodes are required. Each panel is defined by linking two nodes. For example the panel ① links node 1 to node 2, panel ② links node 2 with node 3. Finally, panel ③② links node 32 with node 1. In general, when defining each panel, proceeding from the first node that defined the panel to the second node, the interior is on the left. On the *Set Problem* sheet of the spreadsheet the boundary is defined in the columns Nodes and Panels. For the test problem, there are 32 nodes and panels, and this is stated at the top of the respective column. The coordinates of each node and the nodes that make up the panels are listed.

The definition of the boundary is the most critical part in setting up the problem; errors in this can have a catastrophic effect on the accuracy of the answer. The spreadsheet has a number of methods for checking the boundary data is satisfactory, with appropriate error messages when issues are noted. The button <Check boundary data...> on the *Set Problem* sheet enables the user to carry out a check on the boundary before executing the BEM. This also creates a diagram of the boundary on the *Sketch* sheet, with the chosen interior points plotted within. Below the button is the Panel Centres column that is also completed when the button is activated. This column is populated with the coordinates of the centre of each panel which can be helpful in setting boundary conditions, as the  $x$  and  $y$  values can be readily substituted into an analytic solution of Laplace equation or its derivative on the boundary.

The column following Panels on the *Set Problem* sheet is for setting the boundary condition. The boundary condition is applied to each individual panel and so the number of panels and their indices are copied from the Panels column and are in cells shaded blue. The boundary condition has the discrete form (12), with  $\alpha_i, \beta_i$  and  $f_i$  defined on the panels for  $i = 1, 2, \dots, n$ .

In general, the functions  $\alpha(\mathbf{p}), \beta(\mathbf{p})$  and  $f(\mathbf{p})$  vary on each panel and their representative value is determined as the value at the centre of the panel. Similarly, in setting up test problems with analytic solutions  $\varphi(\mathbf{p})$  and  $\frac{\partial\varphi(\mathbf{p})}{\partial n}$  are functions of  $x$  and  $y$ , determined at the centres of the panels, that are listed in the Panel Centres column.

The Dirichlet boundary condition  $\varphi = 10$  on the left-hand side of the square, for the first eight panels, is achieved by putting  $\alpha_i = 1, \beta_i = 0$  and  $f_i = 10$  for  $i = 1, 2, \dots, 8$ . The Neumann boundary condition on the upper side on the square may be effected by putting  $\alpha_i = 0, \beta_i = 1$  and  $f_i = 0$  for  $i = 9, 10, \dots, 16$ . Similarly for panels 17 to 32, and the values are listed in the Boundary Condition column.

*C. Running the Test Problem and Interpreting the Results*

The buttons on the left activate the computation of the solution. The button <Form BEM Matrices L\_SS, M\_SS, Mt\_SS, L\_PS, M\_PS> activates the computation of the matrices  $L_{SS}, M_{SS}, M_{SS}^t, L_{PS}$  and  $M_{PS}$ , introduced in the previous section. The buttons <Direct Solution> and <Indirect Solution> activate the computation of the solutions via the direct and indirect BEM.

The solutions are listed on the Direct Solution and Indirect Solution sheets. The column phi\_D lists the solution at the chosen interior points and the results from this test are listed in Table I. The boundary solution is also listed on the same sheet in the column 'Solution on S'. This echoes the input boundary condition, with  $\varphi = 10$  on the first eight panels and  $\frac{\partial\varphi}{\partial n} = 0$  on the next eight panels etc. The results show the steady increase in  $\varphi$  from 10 to 20, moving right along the upper and lower sides. The results also show, for example, that  $\frac{\partial\varphi}{\partial n} \approx -10$  on the left side and this can be verified with  $\nabla\varphi = \begin{pmatrix} 10 \\ 0 \end{pmatrix}$  and  $n = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$  and hence, analytically,  $\frac{\partial\varphi}{\partial n} = \nabla\varphi \cdot n = -10$ .

TABLE I  
COMPARISON OF COMPUTED AND EXACT RESULTS FOR SQUARE TEST

Point	Exact	Direct	Indirect
(0.25,0.25)	12.5	12.4957	12.4838
(0.5,0.5)	15	15.0000	14.9836
(0.25, 0.75)	17.5	17.5043	17.4780

With the author's experience in teaching with this spreadsheet, and through guiding students in setting up and running their own problems, it is useful to reinforce the mathematics of the solution of Laplace equation and the calculation of  $\frac{\partial\varphi}{\partial n}$  with this very simple initial test problem. It is important that students are aware that the results are approximations and to compare analytic with numerical solutions. However, there is also a significant opportunity to attach physical meaning to the results, as discussed earlier, and this also prepares the students in applying the spreadsheet to practical problems in their own work. Returning to the heat conduction model, the internal approximations to the temperature are as expected. The results for  $\frac{\partial\varphi}{\partial n}$  also indicate the

flow of heat in and out of the domain, and, since the model is at steady-state, the flow of heat in must be equal to the flow of heat out.

## V. THE BEM FOR AXISYMMETRIC 3D PROBLEMS AND LBEMA

In this section, the BEM is developed for interior and exterior axisymmetric problems. The implementations of the BEM on the LBEMA spreadsheet are outlined and demonstrated with test problems.

### A. The BEM for Axisymmetric 3D Laplace Problems and LBEMA.xlsm

In three dimensions, the same operator notation (5-7) is used, except with  $\Gamma$  representing a whole or partial surface and with  $G$  as the Green's function for the three-dimensional Laplace equation

$$G(\mathbf{p}, \mathbf{q}) = \frac{1}{4\pi r} \quad (15)$$

where  $r$  is the distance between the points  $\mathbf{p}$  and  $\mathbf{q}$ ,  $r = |\mathbf{p} - \mathbf{q}|$ . With these changes of definition, the integral equation reformulation of the interior Laplace problem is the same as those for two-dimensional problems (8) and (9).

In exterior three-dimensional problems, Laplace equation is solved in the domain  $E$  exterior to a surface  $S$ . The integral equation reformulation for exterior problems is similar, but with sign changes. In operator notation, the direct integral equation formulation (3a) and (3b) are as follows:

$$\left\{ \left( M - \frac{1}{2} I \right) \varphi \right\}_S(\mathbf{p}) = \{Lv\}_S(\mathbf{p}) \quad (\mathbf{p} \in S), \quad (16a)$$

$$\varphi(\mathbf{p}) = \{M\varphi\}_S - \{Lv\}_S \quad (\mathbf{p} \in E) \quad (16b)$$

where  $v(\mathbf{q}) = \frac{\partial \varphi(\mathbf{q})}{\partial n_q}$ . The indirect formulation is as follows

$$\varphi(\mathbf{p}) = \{L\sigma\}_S \quad (\mathbf{p} \in SUE), \quad (17a)$$

$$v(\mathbf{p}) = \left\{ \left( M^t - \frac{1}{2} I \right) \sigma \right\}_S \quad (\mathbf{p} \in S). \quad (17b)$$

In order to activate the method, the axisymmetric boundary is approximated by a set of truncated conical panels. Collocation is applied and, with the changes in some definitions as described, the discrete equations for the interior problem are the same as those for the 2D problem (11)-(14).

For the exterior problem, the discrete equivalents of equation (16a) and (16b) are as follows:

$$\left( M_{SS} - \frac{1}{2} I \right) \hat{\varphi}_S = L_{SS} \hat{v}_S, \quad (18a)$$

$$\hat{\varphi}_P = M_{PS} \hat{\varphi}_S - L_{PS} \hat{v}_S. \quad (18b)$$

Similarly, for the indirect method, the discrete equivalent of (17a) and (17b) is

$$\hat{\varphi}_S = L_{SS} \hat{\sigma}_S, \quad (19a)$$

$$\hat{v}_S = \left( M_{SS}^t - \frac{1}{2} I \right) \hat{\sigma}_S, \quad (19b)$$

$$\hat{\varphi}_P = L_{PS} \hat{\sigma}_S. \quad (19c)$$

In the equations above, for the exterior problem,  $\hat{\varphi}_P$  represents the approximation to the solution at the selected exterior points.

### B. Test Problems and Their Implementation on the LBEMA Spreadsheet

The LBEMA spreadsheet has a very similar format as LIBEM2. However, in this case, there are two example sheets for setting the problem, the first is an interior problem and the second is an exterior problem and these sheets have the titles *Set Interior Problem* and *Set Exterior Problem*. On the *Set ... Problem* sheets the four columns Nodes, Panels, Boundary Condition and Interior/Exterior points communicate the test(s) to the spreadsheet program in a similar way. The button <Check Boundary. Find panel centres and sketch> similarly checks the validity of the boundary, completes the Panel Centres column and the *Sketch* sheet. The panels are defined in a similar way as in LIBEM2. However, in this case, the panels are truncated cones, with the two nodes on the generator, with  $(r, z)$  coordinates in Nodes, defining the panel. The two nodal indices define the panel in Panels, with the interior to the right when the nodes are in order.

The buttons on the right activate the computation of the BEM matrices and calculate the direct and indirect boundary element solution on the appropriate sheets. The spreadsheet can solve both the interior and exterior Laplace equation and on the top left corner of the spreadsheet the value TRUE is placed in order to indicate an interior problem and FALSE in order to indicate an exterior problem.

#### Interior Test Problem

The interior test problem is set up on the *Set Interior Problem* sheet in LBEMA. The initial test surface is a cylinder, with the generator linking the upper centre  $(r, z) = (1, 0)$  to the upper edge  $(1, 1)$  to the lower edge  $(1, 0)$  and to the lower centre  $(0, 0)$ .

The boundary condition that is applied is  $\varphi=20$  on the upper surface,  $\varphi=10$  on the lower surface and  $\frac{\partial \varphi}{\partial n} = 0$  on the curved surface. As with the 2D example, there is a straightforward heat conduction analogy and the expected interior solutions are 12.5, 15 and 17.5. The results from LIBEMA are given in Table II.

#### Exterior Test Problem

The exterior test problem is set up on the *Set Exterior Problem* sheet in LBEMA. The test surface is a unit sphere, centred at the origin, defined by 20 panels. In this case, the test problem is set up with the Laplace solution.

$$\varphi(\mathbf{p}) = \frac{1}{r} \text{ for } r \geq 1 \quad (20)$$

where  $r$  is the distance from the origin to the point  $\mathbf{p}$ . A Dirichlet boundary condition  $\varphi = 1$  is applied on the upper

hemisphere. A Neumann boundary condition is applied to the lower hemisphere:

$$\frac{\partial \varphi}{\partial n} = \frac{\partial \varphi}{\partial r} \frac{\partial r}{\partial n} = \frac{-1}{r^2} \cdot 1 = -\frac{1}{r^2} = -1 \text{ (when } r = 1\text{)}.$$

TABLE II  
COMPARISON OF COMPUTED AND EXACT RESULTS FOR CYLINDER TEST

Point (r, z)	Exact	Direct	Indirect
(0.25, 0.25)	12.5	12.4990	12.4980
(0.75, 0.25)	12.5	12.4944	12.4967
(0.25, 0.75)	17.5	17.5012	17.4993
(0.75, 0.75)	17.5	17.5060	17.4899
(0.5,0.5)	15	15.0002	14.9959

The solution is sought at the five exterior points with (r, z) coordinates (0,2), (0,-2), (1,1), (1,-1) and (2,0), at which the exact solutions are  $\varphi = 0.5, 0.5, 0.7071, 0.7071$  and  $0.5$  to four decimal places. The solutions from the direct BEM are listed in Table III.

TABLE III  
COMPARISON OF COMPUTED AND EXACT RESULTS FOR SPHERE TEST

Point (r, z)	Exact	Direct	Indirect
(0, 2)	0.5	0.4988	0.4995
(0, -2)	0.5	0.4987	0.5027
(1, 1)	0.7071	0.7054	0.7061
(1, -1)	0.7071	0.7052	0.7111
(2, 0)	0.5	0.4987	0.5008

## VI. THE BEM ON A SPREADSHEET IN TEACHING VECTOR CALCULUS AND SIMULATION

The spreadsheets LIBEM2 and LBEMA have been used to facilitate teaching and learning in vector calculus and simulation to undergraduate engineering students at the University of Central Lancashire. For both spreadsheets, the students were given an assignment to work through under the guidance of their tutor. The assignment is therefore formative with the student developing their skills and understanding as they carry out their work. In this section, the motivation for and structure of the assignment work is developed, and samples of student work are listed.

### A. Assignment

The assignment imitated the structure of a typical analysis of an applied numerical method and simulation, reported as would be expect in a technical report. The students were asked to develop a test problem using a non-trivial solution of Laplace equation interior to a boundary. Once the test problem has been input to the spread-sheet and executed, a comparison can be made between computed and analytic solutions. Finally, ‘realistic’ boundary conditions, based on a heat conduction problem, are applied and students are expected to attach a practical understanding to the results.

#### 1. Vector Calculus

An important outcome is that students are able to connect the analytical mathematical solution with the computed solution, are able to develop and input the mathematical problem and

interpret the output. Trivial solutions of (the interior) Laplace equation such as  $\varphi = 1$ ,  $\varphi = x$ , or  $\varphi = y$ , or any combination, should be discouraged. Trivial solutions are not significantly testing the software and, more importantly for teaching and learning, are too simple for developing skills in vector calculus. On the other hand, solutions like  $\varphi = xy$  or  $\varphi = x^2 - y^2$  are acceptable and these can be combined with each other and with the trivial solutions to provide a variety. The chosen solution  $\varphi$  may be applied directly as a Dirichlet condition, using the coordinates in the Panel Centres column.

Using the identity  $\frac{\partial \varphi}{\partial n} = \nabla \varphi \cdot \mathbf{n}$ , where  $\mathbf{n}$  is the normal to the boundary, the analytic Neumann boundary condition can be derived. It is possible therefore to have various Dirichlet and Neumann conditions, or to combine them to form the more general Robin condition. Clearly, this is more difficult than applying a Dirichlet condition on the whole boundary. In order to maintain challenge, students may be directed away from the easiest route, alternatively students could be asked to verify the  $\frac{\partial \varphi}{\partial n}$  data given on the solution sheets, comparing the numerical results with the expected mathematical solution.

In showing that the chosen function  $\varphi$  is a solution of Laplace equation and deriving or interpreting  $\frac{\partial \varphi}{\partial n}$  data, the students are demonstrating their skills in vector calculus. As the work is in the context of using computers in engineering, then students may also be expected to use symbolic mathematical calculators to support their analysis.

#### 2. Simulation

From a practical point of view, this assignment is about engineers building personal or collective confidence in software. In carrying out the work of the assignment, the students should observe that the method gives approximations to the test problems, with convergence to the analytic solution as the number of panels increases. Even though the practical heat conduction will normally have no analytic solution, the results should be as reasonably expected.

The test boundary, in 2D or axisymmetric 3D could be of any shape of the students’ choosing. The freedom reinforces the robustness of the underlying method. Various students applying the method to various problems, and all achieving similar sorts of results, also nurtures confidence. This would be much harder to achieve with domain methods (such as the finite element method); the ease of creating the boundary mesh with the BEM enables the educational experience discussed.

Results from the direct and indirect method are given, enabling the comparison of different methods. Different meshes can be applied and thus a connection between mesh size and accuracy can be established.

#### B. Tests Developed by Students

In this section, two extracts from the students’ assignment work is shown. The first example is by a second year student, using LIBEM2 to solve a two-dimensional problem and the second is by a third year student using LBEMA to solve a three-dimensional problem.

### 1. Two-Dimensional Test

A two-dimensional test problem using LIBEM2 is carried out on the shape illustrated in Fig 5. The solution of Laplace equation that forms the basis of the test problem is  $\varphi = x^2 - y^2$ . A Dirichlet boundary condition is applied on all sides, except the upper and lower sides, where a Neumann boundary condition is applied. The computed and exact results are compared in Table IV.

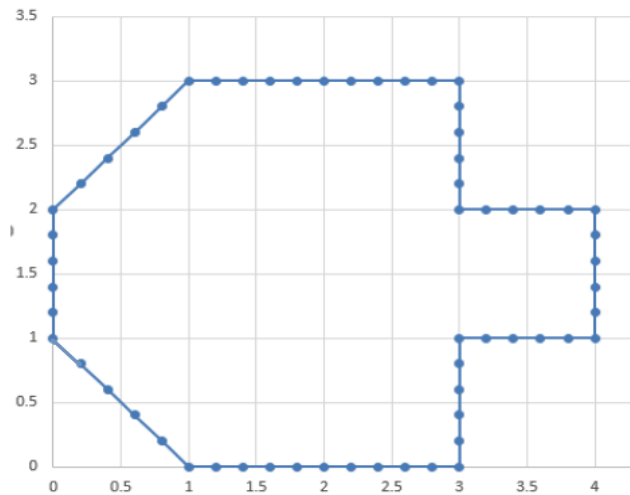


Fig. 5 Two-dimension test shape

TABLE IV  
COMPARISON OF COMPUTED AND EXACT RESULTS FOR 2D TEST

Point	Exact	Direct	Indirect
(3,1.5)	6.75	6.7654	6.7527
(2,2)	0	0.0043	0.0008
(2,1.5)	1.75	1.7560	1.7512
(3.5, 1.5)	10.0	10.0056	10.004
(2,0.5)	3.75	3.7537	3.7509

### 2. Three-Dimensional Test

In the three-dimensional axisymmetric test using LBEMA, the boundary is in the shape of a flask, as shown through a diagram of its generator in Fig. 6. Fig. 6 also shows the location of the interior points at which the solution is sought. In the first test, the exact solution is  $\varphi = r^2 - 2z^2$  and the corresponding Dirichlet boundary condition is applied. The results from this test at a few sample points are given in Table V.

TABLE V  
COMPARISON OF COMPUTED AND EXACT RESULTS FOR FLASK

Point	Exact	Direct	Indirect
2	-12.89	12.888	12.894
7	-11.5	-11.499	-11.500
10	5.289	5.290	5.291

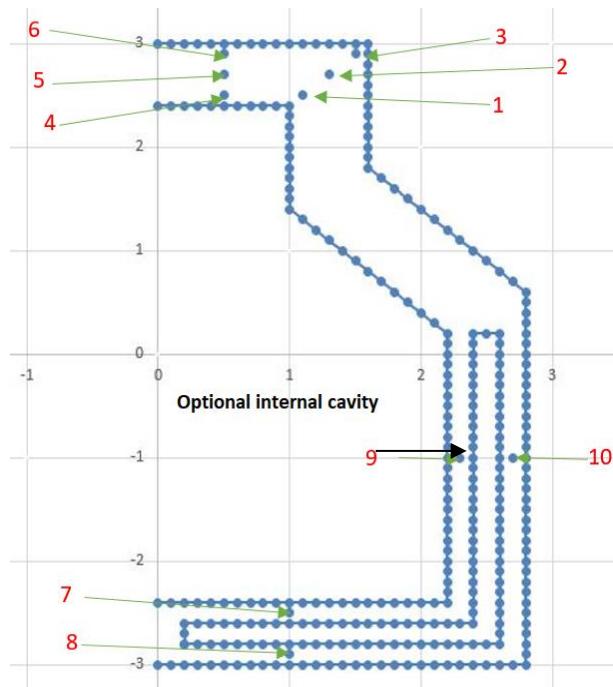


Fig. 6 The generator of the flask

Realistic boundary conditions are then applied to the flask test problem, in which  $\varphi$  is interpreted as the temperature in a steady-state heat conduction problem. The inside boundary is at a temperature of  $3^\circ$  ( $\varphi = 3$ ), and the outside boundary is at a temperature of  $20^\circ$  ( $\varphi = 20$ ). Results with and without the insulated inner cavity ( $\frac{\partial \varphi}{\partial n} = 0$ ) at selected points are given in Table VI.

TABLE VI  
COMPARISON OF RESULTS FOR THE FLASK WITH AND WITHOUT CAVITY

Point	without cavity		With cavity	
	Direct	Indirect	Direct	Indirect
1	10.933	10.933	10.933	10.933
2	17.157	17.157	17.158	17.158
3	19.708	19.708	19.713	19.713
7	5.835	3.002	5.835	3.002
8	17.172	19.986	17.173	19.987
9	6.133	3.001	6.133	3.000
10	17.440	20.000	17.441	20.000

The results show that the direct and indirect method give similar answers. The values at points 1 to 3 show a simple temperature gradient, as we would expect. Finally, the insertion of the cavity has little effect on the results for points 1 to 3, well away from the cavity, however, for points 7-10, either side of it, the results are as we would expect.

### VII. CONCLUSION

Software, in this case built on spreadsheets, enables us to hide complexity. In this case, hiding the complexity of the BEM and all the coding enables us to use in the prosaic stated ambition of an educational aid in vector calculus and simulation. The openness of the spreadsheet environment,



combined with the accessibility of the BEM, supports the teaching and learning.

Computing technology is increasingly being harnessed for educational purposes. The spreadsheets LIBEM2 and LBEMA, considered in this work, are examples of this, linking the worlds of engineering, mathematics and computing.

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