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A new paradigm for the dynamics of the early Universe

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A new paradigm for the dynamics of the early Universe

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Abstract

This paper invokes a new mechanism for reducing a coupled system of fields (including Einstein's equations without a cosmological constant) to equations that possess solutions exhibiting characteristics of immediate relevance to current observational astronomy.

Our approach is formulated as a classical Einstein-vector-scalar-Maxwell-fluid field theory on a spacetime with three-sphere spatial sections. Analytic cosmological solutions are found using local charts familiar from standard LFRW cosmological models. These solutions can be used to describe different types of evolution for the metric scale factor, the Hubble, jerk and de-acceleration functions, the scalar spacetime curvature and the Kretschmann invariant constructed from the Riemann–Christoffel spacetime curvature tensor. The cosmological sector of the theory accommodates a particular single big-bang scenario followed by an eternal exponential acceleration of the scale factor. Such a solution does *not* require an externally prescribed fluid equation of state and leads to a number of new predictions including a current value of the 'jerk' parameter, 'Hopfian-like' source-free Maxwell field configurations with magnetic helicity and distributional 'bi-polar' solutions exhibiting a new charge conjugation symmetry.

An approximate scheme for field perturbations about this particular cosmology is explored and its consequences for a thermalisation process and a thermal history are derived, leading to a prediction of the time interval between the big-bang and the decoupling era. Finally it is shown that field

couplings exist where both vector and scalar localised linearised perturbations exhibit dispersive wave-packet behaviours. The scalar perturbation may also give rise to Yukawa solutions associated with a massive Klein–Gordon particle. It is argued that the vector and scalar fields may offer candidates for 'dark-energy' and 'dark-matter' respectively.

Introduction

The advent of modern satellite technology in observational astronomy has ushered in a new era for astrophysics and research into the fundamental role of gravitation on a large scale. In particular the venerable subject of relativistic cosmology has received a new impetus with the realisation that a number of standard cosmological models may need revision [1, 2]. In this paper we discuss an alternative paradigm that, while retaining many of the most established features of the current standard model, circumvents some of its weaknesses. We invoke a new mechanism for reducing Einstein's field equations (without a cosmological constant term) coupled to fluid matter, vector and scalar fields, to a dynamical system that possesses a class of *simple analytic* solutions for the metric scale factor exhibiting characteristics of immediate relevance to current observational astronomy.

Within the framework of the standard cosmological paradigm there has long been interest in models containing additional vector and/or scalar fields. One that has had considerable impact due to its applications to astrophysics and the problem of hidden matter was pioneered by Milgrom [3] and developed extensively by Bekenstein [4]. Bekenstein's model offers a relativistic post-MOND theory designed to account for a broad range of astrophysical phenomena, including galactic rotation rates, without explicitly introducing *cosmological dark-matter fields*. It is constructed in terms of a constrained dynamic vector field (Π), a dynamic tensor field (g), a dynamic scalar field (ϕ), a non-dynamic field (σ), a Lagrange multiplier field (λ), a phenomenological real-valued function (F) and four coupling parameters K, k, l, G. Matter is introduced following the standard paradigm in terms of ideal multi-fluids, each with their individual equations of state. It is further assumed that each fluid 4-velocity is aligned with the time-like vector field Π and that the 'physical metric' in the fluid stress–energy–momentum tensors is non-trivially related to the tensor field g. Application of the theory to astrophysical phenomena demands wide ranging assumptions and approximations. Although it claims agreement with extra-galactic phenomena, including the lensing of electromagnetic radiation by galaxies and galaxy clusters, concordance with the solar system and binary pulsar tests, it leaves open the question of the need for *cosmological dark matter*.

Our model has more modest aims. It constructs a viable cosmology in terms of four dynamical fields $\mathbb{A}, F, \Psi$, the physical metric tensor g and fewer coupling constants. No phenomenological function is involved and the presence of the Maxwell field F is necessary for developing the electromagnetic sector and essential in identifying the fluctuation solutions $\mathbb{A}^{(1)}, \Psi^{(1)}$ as *dark* elements. The Bekenstein theory follows the standard paradigm of solving the homogeneous, isotropic Einstein equation, involving a multi-fluid with prescribed individual equations of state, for the scale factor. By contrast we propose an alternative paradigm by exploiting an anisotropic ansätz for the vector field $\mathbb{A}$ and a constant value for the scalar field

Ψ leading to a dynamic equation for the scale factor. It remains for future work to ascertain all values of the constants in our theory and to verify whether such a minimalist field system is consistent with the available astrophysical data beyond that considered in this paper.

In section 1 we briefly outline some aspects of the standard approach and draw attention to those weaknesses that have motivated our approach, described in section 2. In section 3 we introduce our general model in terms of a system of field equations for a scalar and vector field interacting with Maxwellian electromagnetism, an anisotropic material fluid and Einsteinian gravitation on spacetime. To develop a particular dynamical cosmological model from this general model we exploit the Maurer–Cartan group structure of the 3-sphere (S^3) to construct a preferred spacetime frame and spacetime metric that shares the homogeneity and isotropy characteristics possessed by those Lemaître–Friedmann–Robertson–Walker (LFRW) metrics with a closed spatial topology.

In section 4 we explicitly construct a class of general analytic solutions for this dynamical system using local charts on spacetime familiar from the LFRW cosmological models. In particular charts, solutions for the LFRW scale factor involve only bounded or hyperbolic trigonometric functions, square roots, a single real parameter and a pair of arbitrary integration constants. We illustrate the different types of Universe evolution that may arise with particular reference to the cosmic time-dependence of the metric scale factor, the Hubble and de-acceleration parameters, the scalar spacetime curvature, the Kretschmann scalar invariant (constructed from the Riemann–Christoffel spacetime curvature tensor and the spacetime metric) once these constants are fixed. We then exhibit the solutions to the Einstein equations for the fluid variables in terms of the scale factor and solutions for the remaining fields in the system. A notable feature of this paradigm is that no additional auxiliary conditions (such as fluid equations of state or Vlasov models) are imposed. By contrast, having fixed all couplings and constants of integration, all cosmological variables have a unique evolution and the fluid variables satisfy algebraic constraints. We interpret such constraints as dynamically induced fluid anisotropic equations of state. Furthermore, because the three-dimensional spatial sections of spacetime with constant time are closed in this model, we show that the vector fields each give rise to configurations with finite total *instantaneous magnetic helicity*. For the Maxwell field this offers a possible interpretation of primordial magnetic fields [5] with closed knotted field lines in an expanding Universe.

In section 5 we use a recent value of the Hubble parameter to single out a particular solution compatible with the current estimate of the *negative* de-acceleration parameter and the existence of an initial singular state for the Universe. The model exhibits a non-uniform exponential expansion for the LFRW scale factor into the future, predicts an age for the Universe and a current value for the unobserved 'jerk' parameter [6].

In section 6 we explore the nature of other solutions by field perturbations about the cosmological solution and show how they may give rise to localised electrically charged domains in space as a prerequisite for scattering processes leading to a thermalisation mechanism. The data used in the cosmological model together with the Planck spectrum is then employed in section 7 to discuss a predicted thermal history leading to an estimate of the time between the big-bang and the decoupling era. Since the scalar and vector field perturbations may provide observational signatures, in section 8 we analyse their first-order partial differential equations (PDEs) for spatially localised solutions in domains where the spacetime is approximately flat. We conclude that the scalar perturbation may give rise to the existence of a massive classical Klein–Gordon particle and both perturbations admit localised dispersive wave-packet solutions.

The concluding section summarises the essential features discussed in the paper.

1. Standard cosmological models

The standard cosmological paradigm is modelled on a class of spacetimes that are spatially homogeneous and isotropic [7, 8]. This class is labelled by an index ($k = 0, \pm 1$) describing spatial global topology with associated symmetry properties and leads to a similarly labelled class of spacetime metrics $\{g_{(k)}\}$. Each metric is supplemented by a choice of a symmetric second-rank tensor $T_{(k)}$ used in Einstein's field equations:

$$\kappa \, \mathrm{Ein}[g_{(k)}] + \Lambda_0 \, g_{(k)} = T_{(k)} \tag{1}$$

where $\Lambda \in \mathbb{R}$ denotes the ubiquitous cosmological constant and $\kappa = c^4/8\pi G$ in terms of the speed of light in vacuo c and the Newtonian gravitational constant G. The Einstein tensor $\mathrm{Ein}[g_{(k)}]$ is defined in terms of the Levi-Civita connection ∇ calculated from $g_{(k)}$ where $\mathrm{Ein}[g_{(k)}] = \mathrm{Ric} - \frac{1}{2}\mathcal{R} g_{(k)}$ in terms of the associated Ricci tensor Ric and Ricci curvature scalar $\mathcal{R}$ associated with $g_{(k)}$. Each $T_{(k)}$ takes the form:

$$T_{(k)} = \sum_{r=1}^{N} \mathcal{T}_{(k)}^{(r)}[g_{(k)}, \rho_{(r)}, p_{(r)}(\rho_{(r)})] \tag{2}$$

where $\mathcal{T}_{(k)}^{(r)}$ often takes the *perfect* fluid form:

$$\mathcal{T}_{(k)}^{(r)} = \left(\rho_{(r)} c^2 + p_{(r)}(\rho_{(r)}) \right) V \otimes V + p_{(r)}(\rho_{(r)}) \, g_{(k)}$$

for some preferred vector field V on spacetime and partial pressure $p_{(r)}(\rho_{(r)})$ specified in terms of some partial mass-energy density $\rho_{(r)} c^2$. Since $\nabla \cdot \mathrm{Ein}[g_{(k)}] = \nabla \cdot g_{(k)} = 0$ identically in Einstein's theory of gravitation and the N fluids are non-interacting (i.e. each $\mathcal{T}_{(k)}^{(r)}$ only depends upon $\rho_{(r)}$), the necessary (Bianchi) consistency condition $\nabla \cdot T_{(k)} = 0$ for all solutions of (1) to satisfy can be achieved by imposing the individual conditions:

$$\nabla \cdot \mathcal{T}_{(k)}^{(r)} = 0 \qquad (\text{for} \quad r = 1, 2, \ldots, N). \tag{3}$$

Many consequences of the standard cosmological paradigm follow from such impositions which are by no means fundamental since the only necessary condition here is

$$\nabla \cdot T_{(k)} = 0.$$

From the assumed homogeneity and spatial isotropy for the model, $g_{(k)}$ can be parametrized by a single real positive scalar function $S_{(k)} : \mathbb{R} \to \mathbb{R}$ for each k, which is a function of some evolution variable in a local spacetime chart. The vector field V is then chosen to be the common unit time-like flow for the composite fluid described by the tensor $T_{(k)}$. Different equations of state for each r are assumed to dominate different epochs in the evolution of the Universe from some initial state to its current state. This evolution is governed by (1) which reduces to a system of independent ordinary differential equations (ODEs) involving $\{S_{(k)}, \rho_{(r)}\}$, given values for the constants $\{\Lambda, N\}$, initial values for $S_{(k)}$ and its first derivative, and the r equations of state $p_{(r)}(\rho_{(r)})$. Equation (3) can, in principal, be integrated to yield $\rho_{(r)}$ in terms of $S_{(k)}$ at the expense of introducing an additional constant of integration for each r.

This program has been pursued extensively in the past in attempts to pin down the appropriate class index k based upon astrophysical data for $\rho_{(r)}$, current values for $S_{(k)}$ and its rate of

change. Furthermore, the discovery of (almost) isotropic black-body thermal electromagnetic radiation has led some authors [9] to invoke additional fields (inflatons) to modify the dynamics described by (1) and (3). Further observations of recent supernova luminosities have raised further concerns that the material content of the Universe, based upon current values of $\rho_{(r)}$, cannot account for the observational data for the current de-acceleration parameter calculated from the variation of $S_{(k)}$ with epoch. This has led to a number of alternative choices for the equations of state $p_{(r)}(\rho_{(r)})$ and the retention of the cosmological constant Λ, sometimes interpreted as a 'Casimir energy' of space.

2. An overview of the new paradigm

In view of these perceived shortcomings we explore an alternative paradigm that *maintains the basic premise of homogeneity and isotropy for the spacetime metric in the model* but dispenses with the imposition of any supplementary fluid equations of state. Instead, we invoke a general model involving field equations for a Maxwell field and a vector and scalar field with specific couplings to each other and gravity, in addition to Einstein's equations involving a material anisotropic fluid. We declare at the outset that the fully coupled system admits a class of $k = +1$ LFRW-type cosmologies given a particular ansätz for the solutions to these field equations.

We assume throughout that the Universe is described by a triple $(\mathcal{M}, g, \Phi)$ where $\mathcal{M}$ is a spacetime endowed with a metric g and Φ is a collection of classical fields. It is implicit in our paradigm that:

Spacetime $\mathcal{M} = \mathcal{I} \times S^3$ where $\mathcal{I}$ denotes an interval of the real line coordinated by t and S^3 is topologically a 3-sphere. The manifold $\mathcal{M}$ is endowed with a $k = +1$ isotropic LFRW covariant Lorentzian metric tensor g in terms of a scale factor $S(t)$ with Φ comprised of a real scalar field Ψ, a real 1-form $\mathbb{A}$ and a real Maxwell 2-form F.

Our particular cosmological model is determined by a system of field equations of the form:

(I) $\mathcal{F}[S, \dot{S}, \ddot{S}, \Psi, \mathbb{A}, F] = 0$

(II) $\kappa \mathrm{Ein}[S, \dot{S}, \ddot{S}] = T[S, \Psi, \mathbb{A}, F, \rho, p_1, p_2, p_3] = \mathcal{T}[S, \Psi, \mathbb{A}, F] + \mathbb{T}[S, \rho, p_1, p_2, p_3].$

Equations (I) and (II) denote a coupled system of second-order partial differential equations involving the fields and the scale factor S and equation (II) denotes Einstein's field equations in terms of the Einstein tensor Ein and total stress–energy–momentum tensor T. This is decomposed into field and matter components: $\mathcal{T}$ and $\mathbb{T}$ respectively, where ρc^2 denotes the cosmological eigen-mass-energy density and p_1, p_2, p_3 the eigen-pressures of the *single* fluid stress-energy-momentum tensor of the matter in the Universe. The paradigm is to solve (I) and (II) as follows:

(i) Construct a suitable ansätz for the fields $\Psi, \mathbb{A}, F$; such that:
(ii) the partial differential field equation (I) are satisfied with $S(t)$ a solution to a second order ordinary differential equation, thereby determining the LFRW metric;
(iii) the Einstein field equation (II) are then solved *algebraically* for the eigen-mass-energy density and eigen-pressures $\{\rho c^2, p_1, p_2, p_3\}$.

An important aspect of this paradigm is that the algebraic relations for ρc^2 and p_1, p_2, p_3, considered as an *induced cosmological anisotropic mechanical equation of state*, are not postulated *a priori*. A consequence of these assumptions is that $\mathrm{Ein}[S, \dot{S}, \ddot{S}]$ contains no term of the form $\Lambda_0 g$ with $\Lambda_0 \in \mathbb{R}$. With a *bona fide* solution to Einstein's field equations, the *total* stress–energy–momentum tensor T has no term of the form $\Lambda_0 g$, implying our model is devoid of a

cosmological constant. Such terms do arise in the *field* stress–energy–momentum tensor $\mathcal{T}$ but are cancelled by terms in the matter stress–energy–momentum tensor $\mathbb{T}$, following from the solutions for the eigenvalues $\{\rho c^2, p_1, p_2, p_3\}$.

3. The general vector-scalar model

3.1. The master field equations

We establish our general field system on the spacetime manifold $\mathcal{M} = \mathcal{I} \times S^3$ where $\mathcal{I}$ is an open subset of the real line and S^3 is topologically a 3-sphere. We endow $\mathcal{M}$ with a covariant Lorentzian metric tensor g, a real scalar field Ψ, a real 1-form $\mathbb{A}$ and a real 2-form F satisfying the master field equations:

$$d \star d\Psi - \frac{1}{2}\kappa_0 \mathcal{U}'(\Psi)\mathbb{A} \wedge \star\mathbb{A} + \frac{1}{2}\Gamma_1 \mathcal{U}'(\Psi)F \wedge F = 0 \tag{4}$$

$$dF = 0, \qquad d \star F + \Gamma_1 \mathcal{U}'(\Psi)\,d\Psi \wedge F = 0 \tag{5}$$

$$d \star d\mathbb{A} + \kappa_0 \mathcal{U}(\Psi) \star \mathbb{A} = 0 \tag{6}$$

where κ_0, Γ_1 are non-zero real coupling constants, with κ_0 having physical dimensions and Γ_1 dimensionless. The map $\mathcal{U} : \mathbb{R} \to \mathbb{R}$ is a potential function and $\star$ denotes the linear Hodge map on forms associated with the general metric g on $\mathcal{M}$. Since, for all *real* fields Ψ and $\mathbb{A}$, equations (5) are invariant under the transformation $A \mapsto A + d\lambda$, where $F = dA$ and λ is an arbitrary 0-form, we identify them with the covariant Maxwell equations for the electromagnetic $U(1)$ gauge invariant 2-form F. Their representation in the SI system will be described below when we discuss the electric current sources in section 6. Equation (6) implies the integrability condition:

$$d(\mathcal{U}(\Psi) \star \mathbb{A}) = 0.$$

Since equations (4)–(6) arise from taking field variations of the action 4-form on $\mathcal{M}$:

$$\mathcal{S}[\Psi, \mathbb{A}, A, g] = \frac{1}{2}d\Psi \wedge \star d\Psi + \frac{1}{2}d\mathbb{A} \wedge \star d\mathbb{A} + \frac{1}{2}\kappa_0 \mathcal{U}(\Psi) \mathbb{A} \wedge \star\mathbb{A}$$
$$+ \frac{1}{2}dA \wedge \star dA + \frac{1}{2}\Gamma_1 A \wedge dA \wedge d(\mathcal{U}(\Psi)) \tag{7}$$

it is straightforward to derive a symmetric stress–energy–momentum tensor $\mathcal{T}$ associated with the fields $\{\Psi, \mathbb{A}, A\}$ from its metric variations:

$$\mathcal{T} = \mathcal{U}(\Psi)\left(d\Psi \otimes d\Psi - \frac{1}{2}g(d\Psi, d\Psi)\,g\right) - \left(i_a d\mathbb{A} \otimes i^a d\mathbb{A} + \frac{1}{2} \star (d\mathbb{A} \wedge \star d\mathbb{A})g\right)$$
$$+ \kappa_0 \mathcal{U}(\Psi)\left(\mathbb{A} \otimes \mathbb{A} - \frac{1}{2}g(\mathbb{A}, \mathbb{A})\,g\right) - \left(i_a dA \otimes i^a dA + \frac{1}{2} \star (dA \wedge \star dA)g\right)$$

where $i_a \equiv i_{X_a}$ denotes the interior contraction operator on forms with respect to any local g-orthonormal basis of vector fields $\{X_a\}$ on $\mathcal{M}$ and $i^a \equiv \eta^{ab} i_{X_b}$ with $\eta^{ab} = \mathrm{diag}(-1, 1, 1, 1)$. Note that the term in (7) containing the coupling Γ_1 does not contribute to the metric variations.

In addition to $\mathcal{T}$ we include the single material fluid symmetric tensor $\mathbb{T}$ parametrized in a local $g - $ orthonormal eigen-cobasis $\{e^a\}$ on $\mathcal{M}$ as

$$\mathbb{T} = \rho c^2 e^0 \otimes e^0 + \sum_{k=1}^{3} p_k e^k \otimes e^k$$

in terms of the real scalar field components $\{\rho, p_1, p_2, p_3\}$ on $\mathcal{M}$. Variational methods for deriving fluid material stress–energy–momentum tensor tensors can be found in [10–12]. The system is closed by adopting $T = \mathcal{T} + \mathbb{T}$ as the source tensor for Einstein's field equations *without* a cosmological constant:

$$\kappa \, \mathrm{Ein}[g] = T. \tag{8}$$

Since κ has physical dimensions of $\mathrm{ML/T}^2$ in the SI system of units it is convenient to assign SI physical dimensions consistently to all tensors in the field system above. To this end, we assign to the covariant rank-two spacetime metric tensor g the physical dimensions of L^2 and write

$$g = -e^0 \otimes e^0 + \sum_{k=1}^{3} e^k \otimes e^k \tag{9}$$

in any g-orthonormal cobase $\{e^a\}$ with elements having physical dimension of length: $[e^a] = \mathrm{L}$. Then the dimensions of the curvature 2-forms $\{R_{ab}\}$ and the curvature scalar

$$\mathcal{R} = \star \left(R_{ab} \wedge \star (e^a \wedge e^b) \right)$$

satisfy $[R_{ab}] = 1$ and $[\mathcal{R}] = \mathrm{L}^2$. Furthermore, these imply $[\mathrm{Ein}] = 1$ yielding dimension of the stress–energy–momentum tensor tensor $[T] = [\mathcal{T}] + [\mathbb{T}] = \mathrm{ML/T}^2$ since $[\kappa] = \mathrm{ML/T}^2$. Bearing in mind that for any p-form β on an n-dimensional manifold: $[\beta] = [\mathrm{d}\beta]$ and

$$[\star\beta] = [\beta] \, \mathrm{L}^{n-2p},$$

we may then consistently assign $[\Psi] = 1, [\mathbb{A}] = [A] = \mathrm{L}$,

$$\kappa_0 = \frac{\zeta_0}{L_0^2}$$

where $[L_0] = \mathrm{L}, [\zeta_0] = 1, [\mathcal{U}(\Psi)] = [\mathcal{U}'(\Psi)] = 1, [\Gamma_1] = 1$. Furthermore the material density ρ has MKS dimensions $\mathrm{M/L}^3$ and the pressures p_1, p_2, p_3 have MKS dimensions $\mathrm{M/LT}^2$ as usual. If all parameters and tensors are specified with values in the SI system of units then the value of the parameter L_0 is unity.

3.2. The cosmological metric ansätz

To construct a particular cosmological model based upon the system of equations (4)–(6) and (8), we need a suitable particular ansätz for $g, \Psi, \mathbb{A}, A$ *and* a preferred eigenbasis for $\mathbb{T}$ defining $\{\rho, p_1, p_2, p_3\}$. To this end, we note that S^3 may be identified with the group manifold $SO(3)$ with an algebra generated by the canonical (2×2) Pauli matrices $\{\sigma^1, \sigma^2, \sigma^3\}$. A group element $U \in SO(3)$ can be coordinated by the three *real* numbers $\{\alpha_1, \alpha_2, \alpha_3\}$ such that:

$$U = \exp\left[\mathrm{i}(\alpha_1\sigma^1 + \alpha_2\sigma^2 + \alpha_3\sigma^3) \right].$$

The three pure imaginary differential 1-forms on $SO(3)$ defined by

$$\widetilde{\Theta}^k = \frac{1}{2}\mathrm{tr}(\sigma^k \, U^{-1}\mathrm{d}U) \qquad (k = 1, 2, 3)$$

satisfy the Maurer–Cartan relations at each point with coordinates $(\alpha_1, \alpha_2, \alpha_3)$:

$$d\widetilde{\Theta}^k = -2i\epsilon^k{}_{lm}\,\widetilde{\Theta}^l \wedge \widetilde{\Theta}^m \qquad (k, l, m = 1, 2, 3) \tag{10}$$

where $\epsilon^k{}_{lm}$ is the Levi-Civita alternating symbol. The expressions for $\{\widetilde{\Theta}^k\}$ in terms of $\{\alpha_1, \alpha_2, \alpha_3\}$ are given in the appendix. These 1-forms offer a preferred cobasis of 1-forms on the group manifold $SO(3) \cong S^3$ and will be used below to define a preferred class of metrics on $\mathcal{M}$. We show in the appendix that a stereographic mapping (A.1) from the coordinates $(\alpha_1, \alpha_2, \alpha_3)$ to a local coordinate chart on S^3 with real coordinates (ξ_1, ξ_2, ξ_3) gives simpler expressions for the cobasis 1-forms on S^3, denoted $\{\Theta^1, \Theta^2, \Theta^3\}$. To define a preferred cobasis of 1-forms $\{e^0, e^1, e^2, e^3\}$ on *spacetime*, we adopt a local spacetime chart with dimensionless coordinates $\{\eta, \xi_1, \xi_2, \xi_3\}$ where $\eta \in \mathcal{I}$ and express the *real* Lorentzian cosmological metric on $\mathcal{M}$ as (9) in terms of

$$e^0 = L_0\,\widetilde{S}(\eta)\,d\eta \qquad \text{and} \qquad e^k = i\,L_0\,\widetilde{S}(\eta)\,\Theta^k \qquad (k = 1, 2, 3) \tag{11}$$

constituting a real *g*-orthonormal cobasis. In this spacetime chart the metric ansätz thereby takes the form:

$$g = L_0^2\,\widetilde{S}^2(\eta)\left[-d\eta \otimes d\eta + \frac{4}{(1 + \xi_1^2 + \xi_2^2 + \xi_3^2)^2}\,(d\xi_1 \otimes d\xi_1 + d\xi_2 \otimes d\xi_2 + d\xi_3 \otimes d\xi_3)\right].$$

A more familiar LFRW form of the metric follows from the coordinate transformation:

$$\left\{\xi_1 = \frac{1}{2}r\sin(\theta)\cos(\phi), \quad \xi_2 = \frac{1}{2}r\sin(\theta)\sin(\phi), \quad \xi_3 = \frac{1}{2}r\cos(\theta)\right\}$$

yielding

$$g = L_0^2\,\widetilde{S}^2(\eta)\left[-d\eta \otimes d\eta + \frac{1}{(1 + \frac{r^2}{4})^2}\,\left(dr \otimes dr + r^2 d\theta \otimes d\theta + r^2\sin^2(\theta)d\phi \otimes d\phi\right)\right]. \tag{12}$$

In this chart, the usual ranges $0 \leqslant r < \infty$, $0 \leqslant \theta \leqslant 2\pi$ and $0 \leqslant \phi < \pi$ for the polar coordinates cover S^3 minus one point. The Maurer–Cartan relations (10) on S^3 imply the following structure equations for the cobasis on $\mathcal{M}$:

$$de^0 = 0 \qquad \text{and} \qquad de^k = \frac{1}{L_0\,\widetilde{S}^2(\eta)}\frac{d\widetilde{S}(\eta)}{d\eta}\,e^0 \wedge e^k - \frac{2}{L_0\,\widetilde{S}(\eta)}\,\epsilon^k{}_{lm}\,e^l \wedge e^m$$

for $k = 1, 2, 3$, which greatly facilitate many of the computations below.

One readily confirms that

$$X_0 = \frac{1}{L_0\,\widetilde{S}(\eta)}\frac{\partial}{\partial\eta} \tag{13}$$

is a unit timelike vector field on $\mathcal{M}$ and that the vector fields $K_j = \widetilde{S}(\eta)\,X_j$ are spacelike Killing vectors on $\mathcal{M}$:

$$\mathcal{L}_{K_j}g = 0 \qquad (j = 1, 2, 3)$$

where $e^a(X_b) = \delta^a_b$ with $a, b = 0, 1, 2, 3$. The vector fields K_j generate the rotation group $SO(3)$ acting on $\mathcal{M}$ confirming that the spacetime metric is spatially isotropic. Spatial homogeneity of any cosmological model follows providing all scalars including components of tensors in

the preferred basis $\{X_a\}$ or dual g-orthonormal cobasis $\{e^a\}$ depend only upon the dimensionless evolution variable η or upon real-valued functions of η.

Many observable astrophysical elements are conventionally specified in terms of 'cosmic time'. We denote this by the dimensionless variable t and relate it to η by the local diffeomorphism $\widehat{\eta} : t \mapsto \eta = \widehat{\eta}(t)$ where

$$\frac{\mathrm{d}\widehat{\eta}(t)}{\mathrm{d}t} = \frac{1}{S(t)} \qquad \text{and} \qquad S(t) = (\widetilde{S} \circ \widehat{\eta})(t) = \widetilde{S}(\widehat{\eta}(t)) = \widetilde{S}(\eta). \tag{14}$$

It is traditional to fix the map $\widehat{\eta}$ so that

$$\widehat{\eta}(t) = \int_0^t \frac{\mathrm{d}t'}{S(t')}. \tag{15}$$

The chain rule is used to relate higher order derivatives of $\widetilde{S}(\eta)$ to derivatives of $S(t)$. Henceforth we distinguish η-derivatives using Lagrange's 'prime' notation and t-derivatives by Newton's 'dot' notation. By slight abuse of notation, we now write elements in the cobasis in terms of the evolution variable t as

$$e^0 = L_0 \, \mathrm{d}t \qquad \text{and} \qquad e^k = \mathrm{i} \, L_0 \, S(t) \, \Theta^k(r, \theta, \phi) \qquad (k = 1, 2, 3) \tag{16}$$

and

$$g = L_0^2 \left[-\mathrm{d}t \otimes \mathrm{d}t + \frac{S^2(t)}{(1 + \frac{r^2}{4})^2} \left(\mathrm{d}r \otimes \mathrm{d}r + r^2 \mathrm{d}\theta \otimes \mathrm{d}\theta + r^2 \sin^2(\theta) \mathrm{d}\phi \otimes \mathrm{d}\phi \right) \right]. \tag{17}$$

We note that singularities of the geometry are likely to occur at any zeroes of S (where the cobase (16) collapses).

4. The cosmological vector-scalar model

4.1. The cosmological vector-scalar ansätz

In terms of the preferred spacetime cobasis (11) induced from the Maurer–Cartan basis $\{i\Theta^k\}$ on S^3, we adopt the ansätz:

$$\mathbb{A} = \Gamma \, e^1 \tag{18}$$

for the 1-form field where $\Gamma \in \mathbb{R}$ is dimensionless and

$$\Psi = \widetilde{\Psi}(\eta)$$

for the scalar field. Thus in addition to a preferred time-like field X_0 describing the global fluid flow the model contains the preferred space-like field X_1 describing the vector field polarisation e^1. The field equation (6) then takes the form:

$$\widetilde{S}''(\eta) = -\widetilde{S}^3(\eta) \, \zeta_0 \, \mathcal{U}(\widetilde{\Psi}) - 4\widetilde{S}(\eta) \tag{19}$$

and (4) becomes

$$\widetilde{\Psi}''(\eta) = -\frac{1}{2}\widetilde{S}^2(\eta) \, \zeta_0 \, \Gamma^2 \, \mathcal{U}'(\widetilde{\Psi}) - 2 \, \widetilde{\Psi}'(\eta) \, \frac{\widetilde{S}'(\eta)}{\widetilde{S}(\eta)}. \tag{20}$$

All elements in these equations are real and dimensionless, including the constant Γ. Once a potential $\mathcal{U}$ is adopted for $\widetilde{\Psi}(\eta)$ and Γ, ζ_0 *are assigned non-zero values*, (19) and (20) constitute a system of non-linear second-order ODEs for $\widetilde{S}(\eta)$ and $\widetilde{\Psi}(\eta)$ amenable to analysis given initial data. Clearly a full analysis of the above coupled system cannot proceed without a specification of the potential $\mathcal{U}$. However since we are interested in particular solutions with interesting physical implications, one may proceed by assuming only that the potential $\mathcal{U}$ has at least one minimum as a function of $\widetilde{\Psi}$, i.e. $\mathcal{U}'(\Psi_0) = 0$ and $\mathcal{U}''(\Psi_0) > 0$ for some real value Ψ_0. Then we observe that the constant field $\widetilde{\Psi}(\eta) = \Psi_0$ is a particular solution to (20) and (19) becomes

$$\widetilde{S}''(\eta) = -\widetilde{S}^3(\eta)\, W - 4\widetilde{S}(\eta) \tag{21}$$

where

$$W \equiv \zeta_0\, \Xi_0 \qquad \text{and} \qquad \Xi_0 \equiv \mathcal{U}(\Psi_0).$$

Furthermore, the terms involving Γ_1 in the field equations do not contribute to the cosmological sector. The general solution to (21) can be written in terms of the Jacobi elliptic sine function sn as:

$$\widetilde{S}(\eta) = \frac{2\sqrt{2}\,\mathcal{K}_2}{\mathcal{P}_1}\; \text{sn}\left(\frac{(\sqrt{2\lambda_0}\,\eta + 2\mathcal{K}_1)\sqrt{2}}{\mathcal{P}_1}, \; \frac{\mathcal{P}_0}{\sqrt{\lambda_0}} \right)$$

where

$$\mathcal{P}_j \equiv \sqrt{(8 - \lambda_0)\mathcal{K}_2^2 + j\lambda_0} \qquad \text{and} \qquad \lambda_0 \equiv W + 8$$

with arbitrary real constants $\mathcal{K}_1, \mathcal{K}_2$. This *general solution* yields a potential class of *physically relevant spacetime cosmographies* for an analysis of Einstein's equations. Since the metric scale factor must be real we can simplify this analysis by first transforming to a new evolution variable.

We proceed by transforming our fields to depend on the cosmic time evolution variable t using (14) with the composition rules:

$$\widetilde{f}'(\eta) \equiv \widetilde{f}'(\widehat{\eta}(t)) = \dot{f}(t)\, S(t)$$
$$\widetilde{f}''(\eta) \equiv \widetilde{f}''(\widehat{\eta}(t)) = S^2(t)\ddot{f}(t) + S(t)\dot{S}(t)\dot{f}(t)$$

relating the functions $f, \widetilde{f}$ of one variable. Then (19) and (20) become for $S(t) \neq 0$:

$$\ddot{S}(t) = -S(t)\,\zeta_0\,\mathcal{U}(\Psi(t)) - \frac{\dot{S}^2(t)}{S(t)} - \frac{4}{S(t)} \tag{22}$$

$$\ddot{\Psi}(t) = -\frac{1}{2}\zeta_0\,\Gamma^2\,\mathcal{U}'(\Psi(t)) - 3\,\dot{\Psi}(t)\,\frac{\dot{S}(t)}{S(t)}.$$

By choosing Ψ_0 as the particular real root of $\mathcal{U}'(\Psi_0) = 0$, (22) becomes

$$\ddot{S}(t) = -S(t)\,W - \frac{\dot{S}^2(t)}{S(t)} - \frac{4}{S(t)}. \tag{23}$$

We note that if $S(t)$ is a real solution to this equation then so is $-S(t)$. Furthermore, for any solution $S(t)$, the expression $a_0 S(t)$ is *not* a solution unless the constant $a_0 = \pm 1$. Solutions to the *non-linear real* equation (23) can be constructed in terms of complex solutions of an *auxiliary equation* for $\Sigma(t)$. This takes the form

$$\ddot{\Sigma}(t) = -2\, W\, \Sigma(t) - 8 \tag{24}$$

with general solution

$$\Sigma(t) = C_{s1} \cos\left(t\sqrt{2\,W}\right) + C_{s2} \sin\left(t\sqrt{2\,W}\right) - \frac{4}{W}. \tag{25}$$

The constants C_{s1}, C_{s2} can be chosen to be real or complex but those of relevance must ensure that

$$S(t) = \pm\sqrt{\Sigma(t)} \tag{26}$$

takes real values for $t \in \mathcal{I}$. From the properties of the hyperbolic functions it is notable that for *arbitrary values* of C_{s1}, C_{s2} and $W < 0$:

$$\lim_{t\to\infty} \mathcal{H}(t) = \sqrt{-\frac{W}{2}}$$

in terms of the *dimensionless* cosmic time Hubble function[4]

$$\mathcal{H}(t) = \frac{\dot{S}(t)}{S(t)}.$$

Furthermore it is evident from the properties of the trigonometric functions that solutions with $W > 0$ are necessary to construct spacetime metrics with a bounded periodic scale factor $S(t)$ (having many real zeroes) whilst those with $W < 0$ may generate real solutions with either unbounded or bounded non-periodic scale factors (with either zero, one or two real zeroes).

Figure 1 displays the profiles of $S(t) = \sqrt{\Sigma(t)}$ that have none, one, two or many real zeroes, obtained by assigning *particular* values to $\{C_{s1}, C_{s2}\}$ and W. Real solutions $S(t)$ with real zeroes indicate the possibility of geometric spacetime singularities. We conclude that solutions of (23) generate possible 'big-bang-big-crunch' cyclic cosmological spacetimes, 'symmetric non-singular' eternal cosmological spacetimes or 'single big-bang eternally expanding' cosmological spacetimes, all with closed spatial universes having the topology of the 3-sphere.

Since the spacetime curvature scalar, $\mathcal{R} = \mathrm{tr}_g(\mathrm{Ric})$, is given in terms of $S(t)$ by

$$\mathcal{R}(t) = -\frac{18}{S^2(t)\, L_0^2} + \frac{6\,W}{L_0^2},$$

solutions where $S(t)$ is constant yield spacetime geometries with a constant curvature scalar. An indication of the presence of singularities in the *geometry* (i.e. those that cannot be eliminated by a change of coordinates) is given where $\mathcal{R}(t)$ becomes unbounded. However, being bounded is not a sufficient condition for the absence of geometric singularities. The Kretschmann scalar $\mathcal{K} = \star(R_{ab} \wedge \star R^{ab})$ offers another invariant that can also locate geometric singularities where it becomes unbounded. In terms of the scale factor of the model under consideration, it is given by

$$L_0^4\, \mathcal{K}(t) = -6\, W^2 - \frac{48\,W}{S^2(t)} - 12\, \mathcal{H}^2(t)\, W - \frac{102}{S^4(t)} - \frac{60\mathcal{H}^2(t)}{S^2(t)} - 12\mathcal{H}^4(t).$$

---

[4] This is related to the *observed* cosmic time Hubble function $\mathcal{H}_{\mathrm{obs}}$ by the relation $\mathcal{H}_{\mathrm{obs}} = c\mathcal{H}/L_0$.

Legend:
- $S(t): \{\, W = 4, \quad C_{s1} = 10, \; C_{s2} = 10 \quad \}$
- $S(t): \{\, W = -1, \; C_{s1} = 1, \quad C_{s2} = 0 \quad \}$
- $S(t): \{\, W = -1, \; C_{s1} = 1, \quad C_{s2} = 1 \quad \}$
- $S(t): \{\, W = -1, \; C_{s1} = 0, \quad C_{s2} = -2i \,\}$
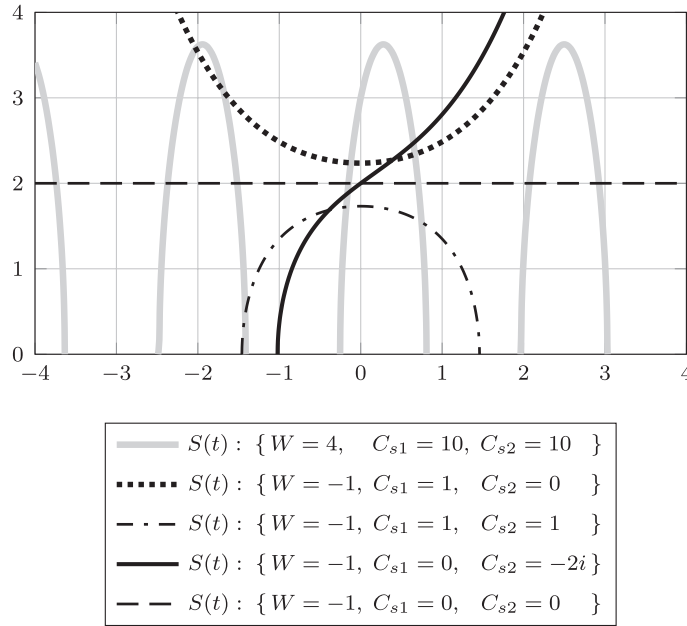- $S(t): \{\, W = -1, \; C_{s1} = 0, \quad C_{s2} = 0 \quad \}$

Figure 1. Typical solutions for the scale factor $S(t)$ satisfying (23).

Recent measurements of type-1a supernova luminosities have also made possible observations of the current value of the dimensionless de-acceleration parameter [13, 14]. In terms of cosmic time, this is defined as

$$q(t) = -\frac{S(t)\,\ddot{S}(t)}{\dot{S}^2(t)}.$$

Using (23) to eliminate the second derivative of the scale factor, this takes the form

$$q(t) = 1 + \frac{4}{\dot{S}^2(t)} + \frac{W}{\mathcal{H}^2(t)} \tag{27}$$

which in our model with $W < 0$, is clearly not manifestly positive and therefore offers the possibility of finding an epoch t in $\mathcal{I}$ where $q(t) < 0$ thereby exploring earlier epochs by retro-diction. For *arbitrary values* of C_{s1}, C_{s2} in the solutions (24) and when $W < 0$ one has the asymptotic behaviour

$$q(t) \simeq -1 + O(\mathrm{e}^{-4Wt})$$

as t approaches infinity. Hence

$$\lim_{t \to \infty} q(t) = -1$$

reminiscent of de Sitter-type solutions. The recent claim that an accelerating phase (i.e. $q(t) < 0$) of the cosmos is consistent with observation has recently led to a number of new cosmological models beyond the standard paradigm.

4.2. The cosmological Maxwell sector

We turn next to the construction of Maxwell fields *in the model with* Ψ *constant* given by $\mathcal{U}'(\Psi_0) = 0$, $\mathcal{U}''(\Psi_0) > 0$. Since the $U(1)$-invariant Maxwell 2-form $F = \mathrm{d}A$ must then satisfy the *linear equation* $\delta F = \delta\,\mathrm{d}A = 0$ on $\mathcal{M}$ where $\delta \equiv \star d\star$ is the divergence operator on p-forms, a general solution could include a *superposition* of singular solutions and non-singular modes generated from *scalar* harmonics on S^3. Typical regular modes were considered earlier by Schrödinger [15] and have subsequently been developed further in the literature [16–18].

Singular solutions would satisfy $\delta F = \delta\,\mathrm{d}A = 0$ on electromagnetic source-free domains of $\mathcal{M}$ free of singularities. In Minkowski spacetime certain distributional Maxwell field solutions serve as models of point particles with electromagnetic multi-pole moments that can interact with regular Maxwell fields. For example stationary electrically charged point particles are modelled by the Coulomb solution where their charge arises as a de-Rham period rather than from a 3-dimensional volume integral of any regular charge density. There exist analogues of such solutions in the spacetime with the metric (12). One readily verifies that in the local chart with coordinates $\{\eta, r, \theta, \phi\}$ the dimensionless Maxwell 2-form:

$$F^I = \frac{Q_0}{4\pi}\left(\frac{1}{4} + \frac{1}{r^2}\right)\mathrm{d}r \wedge \mathrm{d}\eta$$

satisfies $\mathrm{d}F^I = \delta F^I = 0$ where Q_0 is a dimensionless constant and $r \neq 0$. In the preferred frame defined by the unit time-like vector field

$$V \equiv X_0 = \frac{1}{L_0 S(\eta)}\frac{\partial}{\partial\eta},$$

this corresponds to an electromagnetic field with electric and magnetic 1-forms:

$$\mathcal{E}_V^I = i_V F^I = -\frac{Q_0}{4\pi}\left(\frac{1}{4} + \frac{1}{r^2}\right)\frac{\mathrm{d}r}{\widetilde{S}(\eta)L} = \frac{Q_0}{4\pi}\left(\frac{r}{4} + \frac{1}{r}\right)^2\frac{\mathcal{N}}{\widetilde{S}^2(\eta)L_0^2}$$

$$\mathcal{B}_V^I = i_V \star F^I = 0$$

in terms of the space-like unit 1-form

$$\mathcal{N} = \sin(\theta)\cos(\phi)e^1 + \sin(\theta)\sin(\phi)e^2 - \cos(\theta)e^3.$$

At any value of $\eta = \eta_0$ the net electric flux crossing any 2-sphere S^2 surrounding the singularity at $r = 0$ is given by the de-Rham period:

$$\int_{S^2}\star F^I = Q_0.$$

One notes that the norms of F^I and $\mathcal{E}_V^I$ coincide and:

$$|F^I| = \sqrt{\star\left(F^I \wedge \star F^I\right)} = \frac{Q_0}{16\pi\widetilde{S}^2(\eta)L_0^2}\left(\frac{r}{4} + \frac{1}{r}\right)^2$$

implying that both $|F^I|, |\mathcal{E}_V^I|$ diverge as $r \to \infty$. However the anti-podal point of S^3 where $r = \infty$ is not in the local chart. But F^I, g and the coframe $\{e^1, e^2, e^3\}$ are each invariant under the symmetry transformation $\{Q_0 \mapsto -Q_0, r \mapsto \frac{4}{r}\}$ implying that the above local representation of F^I *extends* to a global electrically neutral distributional Maxwell solution with singularities at a pair of antipodal points of S^3 with equal and opposite de-Rham periods at each

pole (see appendix). Since this zero-total electric charge solution on S^3 has a *pair* of spatially distinct singularities we shall designate it an *electric bi-pole* to distinguish it from electrically neutral singular *dipole* solutions that each possess a *single* spatially located singularity.

Similarly Hodge-duality implies the existence of distributional *magnetic bi-pole* solutions of the form:

$$F^{\text{II}} = \frac{M_0}{4\pi} \sin(\theta)\, \mathrm{d}\theta \wedge \mathrm{d}\phi$$

where M_0 is a dimensionless constant and

$$\mathcal{E}^{\text{II}}_V = i_V F^{\text{II}} = 0$$

$$\mathcal{B}^{\text{II}}_V = i_V \star F^{\text{II}} = \frac{M_0}{4\pi}\left(\frac{1}{4} + \frac{1}{r^2}\right)\frac{\mathrm{d}r}{\widetilde{S}(\eta)L_0} = -\frac{M_0}{4\pi}\left(\frac{r}{4} + \frac{1}{r}\right)^2 \frac{\mathcal{N}}{\widetilde{S}^2(\eta)L_0^2}$$

in the chart with coordinates $\{\eta, r, \theta, \phi\}$, with constant dimensionless *magnetic* charge:

$$\int_{S^2} F^{\text{II}} = M_0.$$

Thus F^{II} represents a global magnetically neutral distributional Maxwell solution with singularities at a pair of antipodal points of S^3 with equal and opposite de-Rham periods at each pole and symmetric under the transformation $\{M_0 \mapsto -M_0, r \mapsto \frac{4}{r}\}$. Despite the similarity of these Maxwell field LFRW bi-pole solutions to the Minkowski Coulomb and magnetic monopole solutions in the vicinity of their isolated singularities, neither F^I nor F^{II} is a spatially spherically symmetric solution:

$$\mathcal{L}_{K_j} F^I \neq 0 \quad \text{and} \quad \mathcal{L}_{K_j} F^{\text{II}} \neq 0 \qquad (j = 1, 2, 3)$$

in terms of the spacelike Killing vectors $\{K_j\}$ on $\mathcal{M}$ discussed in section 3.

In the cosmological context that follows we shall content ourselves with a particular *regular source-free superposition* that displays interesting distinct electromagnetic features and consider the particular ansätz:

$$F = d\left(\widetilde{\alpha}(\eta)\, e^1\right) + \widetilde{E}(\eta)\, e^0 \wedge e^1 + \widetilde{B}(\eta)\, e^2 \wedge e^3$$

in terms of the real scalar functions $\widetilde{\alpha}, \widetilde{E}, \widetilde{B}$ of η and return to a discussion of local charge fluctuations in section 6. Now the source free Maxwell equations $\mathrm{d}F = \delta F = 0$ lead to the non-distributional solutions:

$$\widetilde{E}(\eta) = \frac{[C_{k1}\cos(2\eta) - C_{k2}\sin(2\eta)]}{\widetilde{S}^2(\eta)}, \qquad \widetilde{B}(\eta) = \frac{[C_{k1}\sin(2\eta) + C_{k2}\cos(2\eta)]}{\widetilde{S}^2(\eta)}$$

and the second-order ODE:

$$\widetilde{\alpha}''(\eta) - W\,\widetilde{S}^2(\eta)\,\widetilde{\alpha}(\eta) + 2\widetilde{\alpha}'(\eta)\,\frac{\widetilde{S}'(\eta)}{\widetilde{S}(\eta)} = 0$$

for $\widetilde{\alpha}(\eta)$. In terms of the cosmic time variable t:

$$F = d\left(\alpha(t)\, e^1\right) + E(t)\, e^0 \wedge e^1 + B(t)\, e^2 \wedge e^3 \tag{28}$$

where

$$E(t) = \frac{[C_{k1}\cos(2\,\widehat{\eta}(t)) - C_{k2}\sin(2\,\widehat{\eta}(t))]}{S^2(t)}, \quad B(t) = \frac{[C_{k1}\sin(2\,\widehat{\eta}(t)) + C_{k2}\cos(2\,\widehat{\eta}(t))]}{S^2(t)}$$

and $\alpha(t)$ is a solution of

$$\ddot{\alpha}(t) + 3\,\mathcal{H}(t)\,\dot{\alpha}(t) - W\alpha(t) = 0. \tag{29}$$

This is a linear non-autonomous ordinary differential equation for $\alpha(t)$ which can be readily integrated in terms of two arbitrary real constants. Depending upon the sign of W, it resembles the dynamics of an oscillator or an 'anti-oscillator' with time-dependent damping or 'anti-damping' determined by the dimensionless Hubble function $\mathcal{H}(t)$.

The particular solutions involving $\widetilde{E}(\eta)$ and $\widetilde{B}(\eta)$ above generalize the source-free Maxwell solutions found by Kopińksi and Natário [19] in the context of a standard LFRW model involving two non-interacting null fluids. To transform these to a cosmic time variable requires the computation of the integral (15). To simplify our analytic presentation in terms of the evolution variable t, in the following we shall therefore generate a regular Maxwell field from the restricted Maxwell potential 1-form ansätz $A = \alpha(t)\,e^1$ by analogy with the ansätz $\mathbb{A} = \Gamma\,e^1$.

When $\widetilde{S}(\eta_0) \neq 0$ and $\widetilde{\alpha}(\eta_0)$ is regular one can immediately calculate the *instantaneous* Maxwell magnetic helicity 3-form $\widetilde{H}|_{\eta=\eta_0} \equiv A \wedge F|_{\Sigma_3}$ on any 3-dimensional space-like hypersurface Σ_3 defined by $\eta = \eta_0 \in \mathbb{R}$, in terms of $\widetilde{S}(\eta_0)$ and $\widetilde{\alpha}(\eta_0)$:

$$\widetilde{H}\big|_{\eta=\eta_0} = -2\frac{\widetilde{\alpha}^2(\eta_0)}{L_0\,\widetilde{S}(\eta_0)}\,e^1 \wedge e^2 \wedge e^3\bigg|_{\Sigma_3}$$

where Σ_3 is, topologically, the 3-sphere S^3. This gives a *finite* total instantaneous Maxwell magnetic helicity:

$$\int_{S^3} \widetilde{H}\big|_{\eta=\eta_0} = -2\frac{\widetilde{\alpha}^2(\eta_0)}{L_0\,\widetilde{S}(\eta_0)}\,\mathrm{Vol}(S^3) = -4\pi^2\,\widetilde{\alpha}^2(\eta_0)\,L_0^2\,\widetilde{S}^2(\eta_0)$$

since $\mathrm{Vol}(S^3) = \int_{S^3} e^1 \wedge e^2 \wedge e^3\big|_{\Sigma_3} = 2\pi^2\,L_0^3\,\widetilde{S}^3(\eta_0)$.

In a similar way, we may define the *instantaneous* $\mathbb{A}$-magnetic helicity associated with the vector field configuration $\widetilde{\mathbb{A}}(\eta)$ in (18): $\widetilde{\mathbb{H}}|_{\eta=\eta_0} \equiv \mathbb{A} \wedge d\mathbb{A}|_{\Sigma_3}$ and derive, for $\widetilde{S}(\eta_0) \neq 0$, the finite total instantaneous $\mathbb{A}$-magnetic helicity:

$$\int_{S^3} \widetilde{\mathbb{H}}\big|_{\eta=\eta_0} = -2\frac{\Gamma^2}{L_0\,\widetilde{S}(\eta_0)}\,\mathrm{Vol}(S^3) = -4\pi^2\,\Gamma^2\,L_0^2\,\widetilde{S}^2(\eta_0).$$

The existence of non-zero total instantaneous magnetic helicity for the particular solutions where $\mathbb{A}$ and A are proportional to the 1-form e^1 on each constant η hypersurface implies that the corresponding electric and magnetic field lines (i.e. the instantaneous integral curves of the electric and magnetic field vectors), defined by splitting the 2-form field strengths F, $\mathbb{F} = d\mathbb{A}$ respectively into electric and magnetic type 1-form fields relative to the preferred coframe (13), are 'linked' and exhibit a 'Hopfian' knot configuration at each instant $\eta = \eta_0$ [20–22].

It is worth noting that the Maxwell gauge field $A = \widetilde{\alpha}(\eta)\,e^1$ satisfies the condition $\delta A = 0$. This reduces its number of independent physical degrees of freedom to two. Although in general the 1-form $\mathbb{A}$ is not a $U(1)$ gauge field and does not satisfy $\delta\mathbb{A} = 0$, when one restricts to a model where the constant scalar field satisfies $\mathcal{U}'(\Psi_0) = 0$, $\Psi_0 \in \mathbb{R}$, then $\delta\mathbb{A} = 0$ also.

4.3. The cosmological Einstein sector

Next we turn attention to the total stress–energy–momentum tensor T in the Einstein equation (8). Introducing the dimensionless fluid density $\Omega(t) = \rho(t)L_0^2 c^2/\kappa$ and dimensionless fluid pressures $P_k(t) = p_k(t)L_0^2/\kappa$, and substituting the equations $\Psi = \Psi_0$, $A = \alpha(t)\,e^1$, $\mathbb{A} = \Gamma\,e^1$ into T yields the non-zero g-orthonormal T components:

$$T_{00}(t) = -\mathcal{J}^-(t) + \frac{2\alpha^2(t)}{S^2(t)} + \frac{\mathcal{H}^2(t)\alpha^2(t)}{2} + \Omega(t)$$

$$T_{11}(t) = \mathcal{J}^+(t) - \frac{2\alpha^2(t)}{S^2(t)} - \frac{\mathcal{H}^2(t)\alpha^2(t)}{2} - 2\,W + P_1(t)$$

$$T_{22}(t) = -\mathcal{J}^+(t) + \frac{2\alpha^2(t)}{S^2(t)} + \frac{\mathcal{H}^2(t)\alpha^2(t)}{2} - 2\,W + P_2(t)$$

$$T_{33}(t) = -\mathcal{J}^+(t) + \frac{2\alpha^2(t)}{S^2(t)} + \frac{\mathcal{H}^2(t)\alpha^2(t)}{2} - 2\,W + P_3(t)$$

in terms of the dimensionless Hubble function $\mathcal{H}(t)$ and where

$$\mathcal{J}^\pm(t) \equiv \left(\pm\frac{W}{2} - \frac{2}{S^2(t)} - \frac{\dot{S}^2(t)}{2S^2(t)}\right)\Gamma^2 - \frac{\dot{\alpha}^2(t)}{2}\left(1 + 2\,\alpha(t)\mathcal{H}(t)\right).$$

i.e.

$$\mathcal{J}^\pm(t) \equiv \left(\pm\frac{W}{2} - \frac{2}{S^2(t)} - \frac{1}{2}\mathcal{H}^2(t)\right)\Gamma^2 - \frac{\dot{\alpha}^2(t)}{2}\left(1 + 2\,\alpha(t)\mathcal{H}(t)\right).$$

It remains to solve the dynamical system containing (29) and

$$\kappa\,\mathrm{Ein}_t = T_t = \mathcal{T}_t + \mathbb{T}_t \tag{30}$$

where $\{\mathrm{Ein}_t, T_t, \mathcal{T}_t, \mathbb{T}_t\}$ is the form taken by the tensors $\{\mathrm{Ein}, T, \mathcal{T}, \mathbb{T}\}$ with the ansätz (16), (18), (28) and $\Psi = \Psi_0$. This system comprises six ordinary coupled equations for the six variables:

$$\{\,\Omega(t),\,P_1(t),\,P_2(t),\,P_3(t),\,S(t),\,\alpha(t)\,\}.$$

However, any solutions will automatically satisfy the Bianchi identity $\nabla \cdot \mathrm{Ein} = 0$ which implies $\nabla \cdot T = 0$. Hence there must be some relations between the six solutions. For any solutions $\alpha(t), S(t)$, the solutions for the fluid variables are found to be

$$\Omega(t) = \mathcal{J}^-(t) + \frac{(3 - 2\alpha^2(t))}{S^2(t)} + \frac{\mathcal{H}^2(t)}{2}\left(6 - \alpha^2(t)\right)$$

$$P_1(t) = -\mathcal{J}^+(t) + \frac{(7 + 2\alpha^2(t))}{S^2(t)} + \frac{\mathcal{H}^2(t)}{2}\left(2 + \alpha^2(t)\right) + 2\,W$$

$$P_2(t) = \mathcal{J}^+(t) + \frac{(7 - 2\alpha^2(t))}{S^2(t)} + \frac{\mathcal{H}^2(t)}{2}\left(2 - \alpha^2(t)\right) + 2\,W$$

$$P_3(t) = \mathcal{J}^+(t) + \frac{(7 - 2\alpha^2(t))}{S^2(t)} + \frac{\mathcal{H}^2(t)}{2}\left(2 - \alpha^2(t)\right) + 2\,W. \tag{31}$$

Two algebraic relations amongst these fluid variables emerge:

$$\Omega(t) = -W\,\Gamma^2 + \frac{1}{2}\left(P_2(t) - P_1(t)\right) + \frac{3}{S^2(t)} + 3\mathcal{H}(t)^2$$

and

$$P_2(t) = P_3(t).$$

These relations parametrize a *dynamically induced* anisotropic fluid *mechanical* equation of state which is, in general, manifestly dependent on the evolution variable t through the dynamics of the scale factor $S(t)$ and the vector, scalar and Maxwell fields. Note that although the fluid stress–energy–momentum tensor is thereby anisotropic [23], the *total* stress–energy–momentum tensor T_t is not. Spatial anisotropies in the field contributions to T_t cancel those in the fluid contribution $\mathbb{T}_t$ yielding the isotropic Einstein tensor Ein_t with non-zero g-orthonormal components $\mathrm{Ein}_t(X_a, X_b) = \mathrm{Ein}_{ab}(t)$ given by:

$$\mathrm{Ein}_{00}(t) = \frac{3}{L_0^2\, S^2(t)}\left(1 + \dot{S}^2(t)\right) \equiv \frac{3}{L_0^2}\left(\frac{1}{S^2(t)} + \mathcal{H}^2(t)\right)$$

$$\mathrm{Ein}_{kk}(t) = -\frac{1}{L_0^2\, S^2(t)}\left(1 + 2S(t)\ddot{S}(t) + \dot{S}^2(t)\right) \equiv \frac{1}{L_0^2}\left(\mathcal{H}^2(t)(2q(t) - 1) - \frac{1}{S^2(t)}\right)$$

$$(32)$$

for $k = 1, 2, 3$.

5. Testing the cosmological model

In the previous section we showed that the ansätz (18) together with $\Psi(t) = \Psi_0$ at a minimum $\mathcal{U}_0$ of $\mathcal{U}$ was sufficient to reduce the coupled scalar-vector system (4) and (6) to the single condition (23) to determine the scale function $S(t)$, the general solution depending on the constants of integration $\{C_{s1}, C_{s2}\}$ and the parameter

$$W \equiv \zeta_0\,\Xi_0.$$

Solutions for the fluid variables $\Omega(t), P_1(t), P_2(t), P_3(t)$ then follow from the Einstein equation (30) in terms of cosmological solutions for the Maxwell equation (5).

An immediate test of the model can be made by exploiting currently available *measured values* of the Hubble parameter $\widehat{\mathcal{H}}_{\mathrm{obs}}(\widehat{t}_0)$ and the de-acceleration parameter $\widehat{q}_{\mathrm{obs}}(\widehat{t}_0)$ since these are independent of the standard cosmological paradigms based on particular multi-fluid equations of state. The cosmic time evolution variable $\widehat{t}_0$ here, with dimensions of time, is related to the current dimensionless time coordinate t_0 by the relation $\widehat{t}_0 = t_0 L_0/c$. Furthermore, since $[\widehat{\mathcal{H}}] = \mathrm{T}^{-1}$, if one uses SI units of time in seconds, then the parameter $L_0 = 1$ m and our dimensionless Hubble parameter $\mathcal{H}(t_0) = \widehat{\mathcal{H}}_{\mathrm{obs}}(\widehat{t}_0)/c$. Taking $\widehat{\mathcal{H}}_{\mathrm{obs}}(\widehat{t}_0) = 58\,h$ km s^{-1}/Mpc $= 3.243 \times 10^{-18}\,h$ s^{-1} yields $\mathcal{H}(t_0) = 1.082 \times 10^{-26}\,h$ with $|h| = 0.58$ [8] while the dimensionless value of $\widehat{q}_{\mathrm{obs}}(\widehat{t}_0) = q(t_0) = -0.56$ is compatible with current observation.

Since (27) implies the de-acceleration parameter $q(t) > 0$ for $W > 0$, contrary to current observation, we will concentrate attention on solutions $S(t)$ of (23) with $W < 0$. Particular solutions are then fixed by specifying values for $W, S(0)$ and $\dot{S}(0)$ since there is no loss of generality by labelling $t_0 = 0$. The values of $S(0)$ and $\dot{S}(0)$ are then in turn furnished in terms of values for $\mathcal{H}(0)$ and $q(0)$ for some $W < 0$. Evaluating (23) at $t = 0$ yields the following value for $S(0)$:
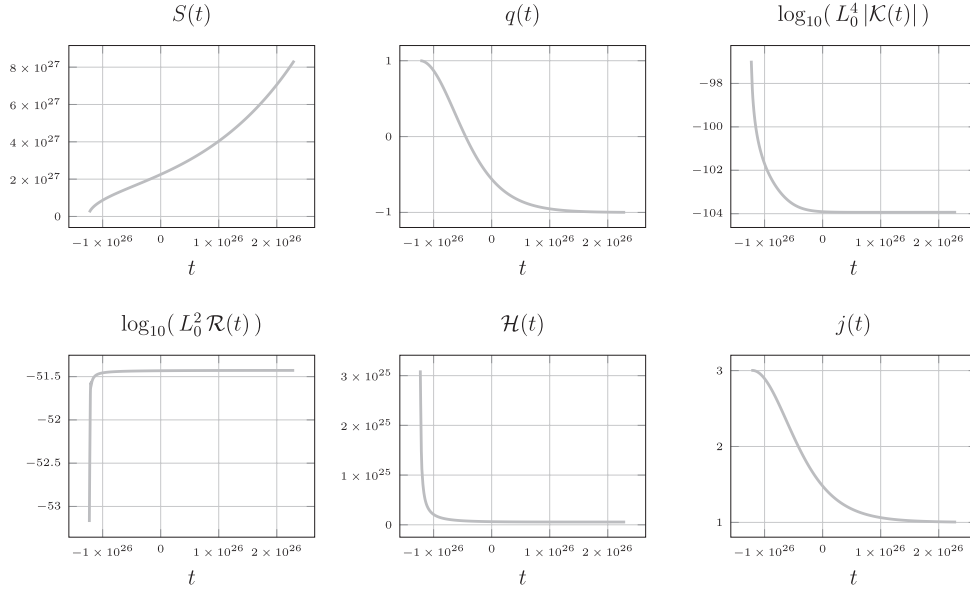
Figure 2. Astrophysical model predictions based on current $(t = 0)$ values of the dimensionless Hubble parameter $\mathcal{H}(0)$ and the de-acceleration parameter $q(0)$.

$$S(0) = \frac{2}{\mathcal{H}(0) \sqrt{q(0) + \Delta^2 - 1}} \tag{33}$$

where $W = -\mathcal{H}(0)^2 \Delta^2$ with $\Delta \in \mathbb{R}$. Then

$$\dot{S}(0) = \mathcal{H}(0) S(0)$$

and the solution $S(t)$ is fixed and real for any $\Delta^2 > 1 - q(0)$.

Figure 2 shows cosmological characteristics based on the solution[5] obtained with $\Delta = 1.257$, giving $S(0) = 2.25 \times 10^{27}$ and retro-dicts the location of a single zero of S at $t = -1.245 \times 10^{26}$ together with a non-uniformly exponentially increasing value of $S(t)$. The curves for $\mathcal{K}$ and $\mathcal{R}$ clearly show that the zero of S is a geometric singularity so one may conclude that the model fitting $\mathcal{H}$ and q with $\Delta = 1.257$ at the current epoch yields an age of the Universe to be around 13.2×10^9 years. The second figure displays the past and future behaviour of the de-acceleration function $q(t)$ while the fifth figure displays the past and future behaviour of the dimensionless Hubble function $\mathcal{H}(t)$. Although there is no data for the current value of the dimensionless 'jerk' function:

$$j(t) = \frac{S(t)^2 \, \dddot{S}(t)}{\dot{S}(t)^3},$$

the model predicts $j(0) = 1.480$ and the last plot in figure 2 displays its past and future behaviour.

---

[5] In terms of the complex constants in (26) this data corresponds to the values: $C_{s1} = 5.002 \times 10^{54}$, $C_{s2} = -5.700\text{i} \times 10^{54}$ and $W = -6.223 \times 10^{-53}$.

In general, when $W < 0$ one has:

$$\lim_{t \to \infty} j(t) = 1$$

for *arbitrary values* of C_{s1}, C_{s2} in the solutions of (24).

Although all these cosmographic features are mathematically independent of the Einstein equations [24], they are consistent with a single 'big-bang' scenario undergoing eternal non-uniform exponential accelerated expansion. The Einstein equations in turn then predict the behaviour of the density and pressures in the fluid stress–energy–momentum tensor compatible with the Maxwell, scalar and vector field stress–energy–momentum tensors, determined from their associated field equations. It is worth noting at this stage that none of the particular solutions that follow from the equations for $\alpha(t), S(t), \Psi(t), \mathbb{A}$, give rise to a term in T of the form $\Lambda_0 g$ with Λ_0 constant and g the spacetime metric tensor (i.e. to an effective cosmological *constant*).

It is also of interest to examine what the above cosmological solution implies for the various local 'energy conditions' that feature in a number of theoretical criteria for general relativistic stress–energy–momentum tensors. These are based on inequalities that involve the eigenvalues of such symmetric tensors. Thus one may express the right hand side of the Einstein equation (30) as

$$T_t = \frac{\widehat{\Omega}(t)}{L_0^2} \, e^0 \otimes e^0 + \frac{1}{L_0^2} \sum_{k=1}^{3} \widehat{P}_k(t) \, e^k \otimes e^k.$$

in terms of the *eigen-mass-energy density* $\widehat{\Omega}(t)$ and *eigen-pressures* $\widehat{P}_k(t)$ and note that the total mass-energy of the interacting system includes the mass-energy of the fields as well as the material fluid. The eigenvalues of T_t following simply from the eigenvalues of Ein_t displayed in (32). Cosmological stress–energy–momentum tensors that satisfy these local 'energy conditions' at cosmic time t obey the constraints on the left in figure 3 and the figures on the right illustrate the behaviour of the expressions in the Dominant, Strong, Weak and Null energy inequalities as a function of dimensionless cosmic time over an interval, starting from the big-bang and including the current epoch ($t = 0$). Over this interval all inequalities are satisfied except the Strong Energy Condition. Opinion seems divided on the significance of this condition in constraining cosmological models [25].

6. A scalar and vector field fluctuation model

With a dynamic model that yields an evolving scale factor one may construct a recursive framework for analysing the *fluctuation and thermal history* of the matter and *field* content of the Universe consistent with the solutions (31) to Einstein's equation (8) without a cosmological constant. This may offer some insight into the nature of 'dark-matter' [26].

It has been shown in section 4 that the ansätz (16), defining the metric g, and the ansätz (18) for $\mathbb{A}$ yield analytic solutions to the equations (4) and (6) provided the constant solution $\Psi = \Psi_0$ satisfies $\mathcal{U}'(\Psi_0) = 0$ and the metric scale factor $S(t)$ is a solution of the non-linear ODE (23), with the general solution given by (26). Any particular solution of (29) then generates a source free Maxwell field that completes a cosmological solution to the coupled field system (4)–(6) compatible with Einstein's equations.

In section 5 particular solutions $S(t)$ to (23) have been shown to describe distinct cosmological spacetimes including those possessing a *single* initial singularity in the spacetime geometry. However, given $\mathcal{U}(\Psi)$, the fully coupled *general system* (4)–(6) and (8) has other
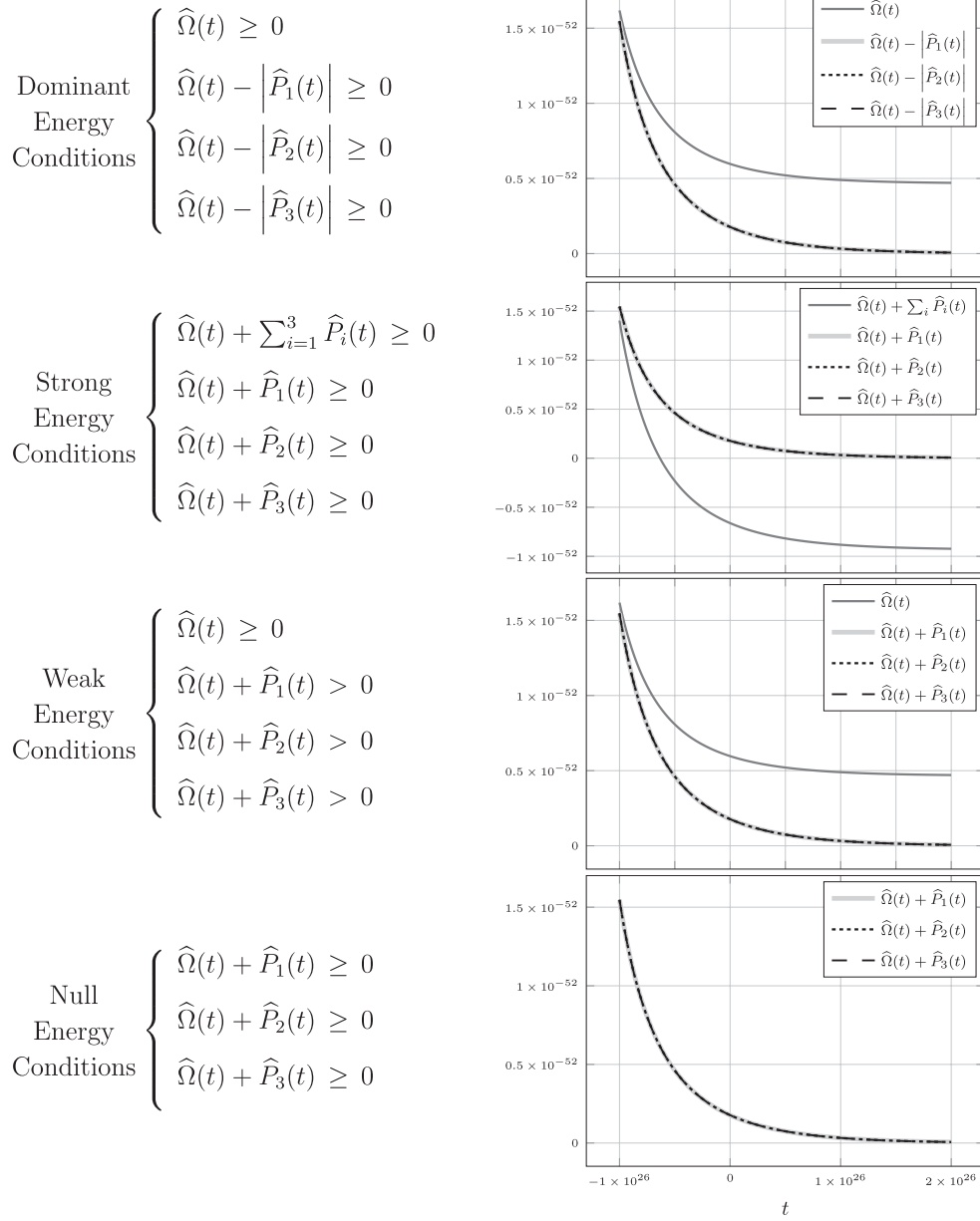
Dominant Energy Conditions
$$\begin{cases} \widehat{\Omega}(t) \geq 0 \\[6pt] \widehat{\Omega}(t) - \left|\widehat{P}_1(t)\right| \geq 0 \\[6pt] \widehat{\Omega}(t) - \left|\widehat{P}_2(t)\right| \geq 0 \\[6pt] \widehat{\Omega}(t) - \left|\widehat{P}_3(t)\right| \geq 0 \end{cases}$$

Strong Energy Conditions
$$\begin{cases} \widehat{\Omega}(t) + \sum_{i=1}^{3} \widehat{P}_i(t) \geq 0 \\[6pt] \widehat{\Omega}(t) + \widehat{P}_1(t) \geq 0 \\[6pt] \widehat{\Omega}(t) + \widehat{P}_2(t) \geq 0 \\[6pt] \widehat{\Omega}(t) + \widehat{P}_3(t) \geq 0 \end{cases}$$

Weak Energy Conditions
$$\begin{cases} \widehat{\Omega}(t) \geq 0 \\[6pt] \widehat{\Omega}(t) + \widehat{P}_1(t) > 0 \\[6pt] \widehat{\Omega}(t) + \widehat{P}_2(t) > 0 \\[6pt] \widehat{\Omega}(t) + \widehat{P}_3(t) > 0 \end{cases}$$

Null Energy Conditions
$$\begin{cases} \widehat{\Omega}(t) + \widehat{P}_1(t) \geq 0 \\[6pt] \widehat{\Omega}(t) + \widehat{P}_2(t) \geq 0 \\[6pt] \widehat{\Omega}(t) + \widehat{P}_3(t) \geq 0 \end{cases}$$

Figure 3. Plots displaying constraints for the Dominant, Strong, Weak and Null Energy Conditions respectively. Some curves are indistinguishable in the plots.

solutions that can break the spatial homogeneity and isotropy of the particular LFRW metric solution. Such solutions may, given additional physical data, ultimately account for the formation of plasma states in the early Universe, needed to thermalise matter and electromagnetic radiation, and for the formation of the elements, stars and galaxies during later epochs of its evolution.

A tentative approximation scheme, adopted below, is to generate a sequence of linearised perturbations of the equations (4)–(6) about the exact cosmological solution described in section 5 in terms of the scale factor (26) in the LFRW metric. This particular scheme assumes the evolving perturbations have a negligible back-reaction on the ambient LFRW gravitational field. This is reasonable for field fluctuations that are localised in small regions of space. In section 8 we will also describe analytically, field perturbations that are *spatially localised* during epochs were the ambient gravitational field is itself neglected, i.e. where the dominant physics under consideration takes place in a background Minkowski spacetime.

Working to order ϵ^2 in the dimensionless perturbed fields, let:

$$
\begin{aligned}
\Psi &= \Psi^{(0)} + \epsilon\Psi^{(1)} + \epsilon^2\Psi^{(2)} \\
\mathbb{A} &= \mathbb{A}^{(0)} + \epsilon\mathbb{A}^{(1)} + \epsilon^2\mathbb{A}^{(2)} \\
A &= A^{(0)} + \epsilon A^{(1)} + \epsilon^2 A^{(2)}
\end{aligned}
\tag{34}
$$

with norms $|\Psi^{(j)}| \leqslant 1$, $|\star(\mathbb{A}^{(j)} \wedge \star\mathbb{A}^{(j)})| \leqslant 1$, $|\star(A^{(j)} \wedge \star A^{(j)})| \leqslant 1$ for $j = 1, 2$. Up to this maximum order it is sufficient to write the potential $\mathcal{U}$ in the form of a truncated Taylor series about the constant value Ψ_0 where it has a *local minimum*:

$$
\mathcal{U}(\Psi) = \Xi_0 + \frac{1}{2}\Xi_1(\Psi - \Psi_0)^2 + \frac{1}{6}\Xi_2(\Psi - \Psi_0)^3
$$

with

$$
\Xi_0 \equiv \mathcal{U}(\Psi_0), \qquad \mathcal{U}'(\Psi_0) = 0, \qquad \Xi_1 \equiv \mathcal{U}''(\Psi_0) > 0 \qquad \text{and} \qquad \Xi_2 \equiv \mathcal{U}'''(\Psi_0).
$$

In terms of the summands in the expansion of Ψ above, it follows that to order ϵ^2:

$$
\mathcal{U}(\Psi) = \Xi_0 + \frac{1}{2}\Xi_1(\Psi^{(1)})^2\,\epsilon^2, \qquad \mathcal{U}'(\Psi) = \Xi_1\Psi^{(1)}\epsilon + \left(\Xi_1\Psi^{(2)} + \frac{1}{2}\Xi_2(\Psi^{(1)})^2\right)\epsilon^2.
\tag{35}
$$

Substituting the expansions (34) and (35) into the PDE equations (4)–(6) yields the zeroth ϵ-order PDE system:

$$
\begin{aligned}
&d \star d\Psi^{(0)} = 0 \\
&d \star d\mathbb{A}^{(0)} + \frac{W}{L_0^2}\star\mathbb{A}^{(0)} = 0 \\
&dF^{(0)} = 0, \qquad d\star F^{(0)} = 0,
\end{aligned}
\tag{36}
$$

the first ϵ-order PDE system:

$$
d \star d\Psi^{(1)} + \frac{\Xi_1}{2}\left(\Gamma_1 F^{(0)} \wedge F^{(0)} - \frac{\zeta_0}{L_0^2}\mathbb{A}^{(0)} \wedge \star\mathbb{A}^{(0)}\right)\Psi^{(1)} = 0
\tag{37}
$$

$$
d \star d\mathbb{A}^{(1)} + \frac{W}{L_0^2}\star\mathbb{A}^{(1)} = 0
\tag{38}
$$

$$
dF^{(1)} = 0, \qquad d\star F^{(1)} = 0,
\tag{39}
$$

and the second ϵ-order PDE system:

$$
d \star d\Psi^{(2)} + \frac{\Xi_1}{2}\left(\Gamma_1 F^{(0)} \wedge F^{(0)} - \frac{\zeta_0}{L_0^2}\mathbb{A}^{(0)} \wedge \star\mathbb{A}^{(0)}\right)\Psi^{(2)} = J
\tag{40}
$$

$$d \star d\mathbb{A}^{(2)} + \frac{W}{L_0^2} \star \mathbb{A}^{(2)} = \mathbb{J} \tag{41}$$

$$dF^{(2)} = 0, \qquad d \star F^{(2)} = \mathcal{J} \tag{42}$$

where

$$F^{(j)} = dA^{(j)}, \qquad (j = 0, 1, 2)$$

and

$$J \equiv + \left(\frac{\zeta_0 \Xi_1}{L_0^2} \mathbb{A}^{(0)} \wedge \star\mathbb{A}^{(1)} - \frac{\Gamma_1 \Xi_1}{2} \left(F^{(1)} \wedge F^{(0)} + F^{(0)} \wedge F^{(1)} \right) \right) \Psi^{(1)}$$

$$+ \left(\frac{\zeta_0 \Xi_2}{4L_0^2} \mathbb{A}^{(0)} \wedge \star\mathbb{A}^{(0)} - \frac{\Gamma_1 \Xi_2}{4} F^{(0)} \wedge F^{(0)} \right) (\Psi^{(1)})^2$$

$$\mathbb{J} \equiv -\frac{\Xi_1 \zeta_0}{2L_0^2} (\Psi^{(1)})^2 \star \mathbb{A}^{(0)}$$

$$\mathcal{J} \equiv -\Gamma_1 \Xi_1 \Psi^{(1)} d\Psi^{(1)} \wedge F^{(0)} = -\Gamma_1 \Xi_1 d \left(\frac{1}{2}(\Psi^{(1)})^2 \wedge F^{(0)} \right).$$

One sees from these equations that, given $\Psi^{(0)}, \mathbb{A}^{(0)}$ and $F^{(0)}$, the scheme generates a sequence of PDE systems for $\Psi^{(j)}, \mathbb{A}^{(j)}$ and $F^{(j)}$. With $\Psi^{(0)} = \Psi_0$, $\mathcal{U}'(\Psi_0) = 0$ and $\mathbb{A}^{(0)} = \Gamma e^1$, the zeroth-$\epsilon$ order system is satisfied by solving the *source-free* Maxwell system (36). Thus these zeroth ϵ-order equations are those used in sections 1–3 to establish a viable cosmological model when combined with the Einstein equations. The first ϵ-order field modifications to this exact cosmological solution (37)–(39) are given by solutions to an uncoupled, *homogeneous*, (source-free) second-order PDE system. By contrast, at the next ϵ-order, the perturbations (40)–(42) satisfy uncoupled *inhomogeneous* equations with sources generated by solutions to the lower ϵ-order systems.

Of particular note is the perturbed Maxwell system (42) for the perturbation $F^{(2)}$ containing the manifestly conserved electric current 3-form $\mathcal{J}$:

$$d\mathcal{J} = 0.$$

Since the SI system of units accommodates quantities with the dimensions q of electric charge, it is convenient at this point to express (42) as

$$d\widehat{F}^{(2)} = 0, \qquad d \star \widehat{F}^{(2)} = \widehat{\mathcal{J}}$$

in terms of the dimensioned forms:

$$\widehat{F}^{(2)} = \frac{q_0}{L_0 \epsilon_0} F^{(2)}, \qquad \widehat{\mathcal{J}} = \frac{q_0}{L_0 \epsilon_0} \mathcal{J}$$

where the real constant q_0 carries the dimensions of charge and ϵ_0 is the permittivity of free space with SI dimensions $q^2 T^2/M L^3$. Then in units with charge q measured in Coulombs, Newtonian force (N) measured in Newtons and length (L) in metres, $\epsilon_0 = 8.8542 \times 10^{-12} q^2 N^{-1} L^{-2}$. At any spacetime point, in any local frame of reference defined by a unit time-like vector field V on $\mathcal{M}$, the charge density $\rho^{(V)}$ in Coulombs per cubic metre is then given by $-i_V \star \widehat{\mathcal{J}}$ and the associated electric current 2-form $J^{(V)}$ in amps per square metre is given by $c\, i_V\, \widehat{\mathcal{J}}$ at that point.

Since $F^{(0)}$ and $F^{(1)}$ are source free solutions, assumed regular and singularity free, the net electric charge $Q[\Sigma_3]$ in any 3-dimensional spacelike domain $\Sigma_3 \subset S^3$ with boundary $\partial\Sigma_3 \neq 0$, is:

$$Q[\Sigma_3] = \int_{\Sigma_3} \widehat{\mathcal{J}} = \int_{\Sigma_3} d \star \widehat{F}^{(2)}.$$

This implies, by Stokes' theorem, that

$$Q[\Sigma_3] = \int_{\partial\Sigma_3} \star\widehat{F}^{(2)}$$

is the net electric flux emanating from the 2-surface $\partial\Sigma_3$ bounding Σ_3. The value of $Q[\Sigma_3]$ may be positive or negative, implying the existence of localised charge fluctuations of different polarities in different regions. However if one integrates J over an entire *closed* space-like hypersurface, topologically S^3, at any instant, then $Q[S^3] = 0$ since the 3-sphere has no boundary across which the electric flux can cross ($\partial S^3 = 0$). Thus the *total electric charge* (induced by the scalar fields $\Psi^{(1)}$ and $F^{(0)}$ at order ϵ^2) must be zero. In any cosmological model the presence of locally charged regions in space seems to be an essential prerequisite for providing scattering processes in a charged particle-field interpretation of the thermalisation mechanism between matter and radiation prior to any decoupling era where ionisation ceases.

7. Thermal history within the new paradigm

The existence of a high temperature state of the early Universe has achieved almost universal recognition based on the accurate measurements of an extra-galactic omni-directional Planck distribution of electromagnetic radiation with a maximum intensity in the microwave spectrum. Assuming that such a radiation distribution has evolved with the expansion of the Universe without change of shape since it escaped from the opaque plasma state, any cosmological model should accommodate the high precision COBE and Planck data [27]. If

$$\mathcal{E}[\Sigma_3] = V[\Sigma_3] \int_0^\infty \widehat{U}_\omega(\mathsf{T}) \, d\omega$$

denotes the total thermal energy in any spatial domain Σ_3 with volume $V[\Sigma_3]$ in an early plasma state then it has a Planck spectrum with temperature T if

$$\widehat{U}_\omega(\mathsf{T}) = \frac{\hbar\omega^3}{\pi^2 c^3} \frac{1}{\exp\left(\frac{\hbar\omega}{k\mathsf{T}}\right) - 1}$$

where $\hbar, k$ are the reduced Planck and Boltzmann constants respectively. Assuming that each harmonic frequency component in this distribution transports electromagnetic energy along a null geodesic of an isotropic background LFRW metric g, one may exploit the ray-optics approximation for harmonic solutions to the source-free Maxwell equations [7] to relate the frequencies or wavelengths of these components made by observers at different spacetime events p. In a cosmology with a LFRW metric a key feature is the existence of a preferred unit time-like vector *field V* that can be used to define the angular frequency $\omega_p = 2\pi \nu_p$ and wavelength $\lambda_p = 2\pi c/\omega_p$ of such solutions at any event p. Then

$$\frac{\nu_e[V_e]}{\nu_o[V_o]} = \frac{g(V_e, K_e)}{g(V_o, K_o)} \equiv 1 + z_{o,e},$$

in terms of the redshift parameter $z_{o,e}$. Here e denotes an emission event where the tangent to the null geodesic is the null vector K_e of a harmonic Maxwell wave with frequency ν_e in the cosmic frame V_e. A similar statement refers to the observation of the frequency ν_o at an observation point o in the causal-future of e. With $V = \frac{1}{L_0}\frac{\partial}{\partial t}$ it follows from (17) that

$$\frac{\nu_e}{\nu_o} = \frac{S(t_0)}{S(t_e)} \tag{43}$$

and

$$z_{t_0,t_1} = \frac{S(t_0)}{S(t_1)} - 1 = \sqrt{\left|\frac{\Sigma(t_0)}{\Sigma(t_1)}\right|} - 1 \tag{44}$$

in terms of the general solution (25). Defining

$$U_\lambda(\mathsf{T}) = \widehat{U}_\omega(\mathsf{T})\big|_{\omega = \frac{2\pi c}{\lambda}}$$

and the change of variable

$$y \equiv \frac{2\pi\hbar c}{\lambda k \mathsf{T}}, \tag{45}$$

the distribution $U_\lambda(\mathsf{T})$ in wavelength becomes

$$U_y(\mathsf{T})\,d\lambda = \frac{k^5 \mathsf{T}^5}{2\hbar^4\pi^3 c^4}\,\mathcal{F}(y)\,d\lambda \tag{46}$$

in terms of the dimensionless universal function

$$\mathcal{F}(y) = \frac{y^5}{e^y - 1}. \tag{47}$$

Since the COBE and Planck data measure the *variation* of $U_y(\mathsf{T})$ with λ_o with a definite temperature $\mathsf{T}_o \simeq 2.73$ K (after correcting for the detector's motion relative to the frame V_o) in agreement with (46), it follows from (45) that y remains independent of epoch after decoupling, i.e.

$$\frac{\lambda_o}{\lambda_e} = \frac{\mathsf{T}_e}{\mathsf{T}_o}$$

and hence from (43):

$$S(t_e) = S(t_0)\frac{\mathsf{T}_o}{\mathsf{T}_e}.$$

If the scale factor $S(t)$ is continuous monotonic-increasing with t and $0 < \mathsf{T}_o/\mathsf{T}_e < 1$ then t_e lies at a time between the initial singularity event and the current epoch t_0. An estimate of the temperature T_e when the Universe first became transparent to thermal radiation would give an estimate of the ratio $S(t_e)/S(t_0)$ since T_o is measured. In the standard cosmological model the value of $S(t_0)$ is estimated from the current densities of matter and radiation in a solution of the Friedman equations. In the cosmological model discussed in this paper it is fixed by the solution (26) using the current values of $\mathcal{H}(t_0)$ and $q(t_0)$ that fix the constants C_{s1}, C_{s2} and W assuming $\Delta = 1.257$. If one assumes a value $\mathsf{T}_e \simeq 4000$ K for the temperature at decoupling, based on a model for the physics of thermalisation in the hot plasma, together with a value $S(t_0) = 2.25 \times 10^{27}$ defined by (33) one may estimate the value for the scale factor at decoupling to have been $S(t_e) \simeq 1.54 \times 10^{24}$. Finally, denoting by t_{BB} the dimensionless cosmic

time of the geometric singularity, then the dimensionless time interval between the big-bang and the decoupling era $t_e - t_{BB}$, is determined to be 7.78×10^{19}, i.e. 8.23×10^3 years.

8. Extending the fluctuation model with perturbative scalar and vector Minkowski modes

In the current epoch, with $t = t_0$, many physical phenomena can be described in terms of field theories on a background flat Minkowski spacetime. If one introduces new coordinates $\widehat{T}$ and $\widehat{R}$ (with dimensions of time and length respectively) by the substitutions $t = c\,\widehat{T}/L_0$ and $r = \widehat{R}/S(t_0)L_0$ then in coordinate domains where $\widehat{R} \ll 2S(t_0)L_0$ the LFRW metric (17) is well approximated by the Minkowski metric in standard spherical polar spatial coordinates:

$$g \approx -c^2\, d\widehat{T} \otimes d\widehat{T} + d\widehat{R} \otimes d\widehat{R} + \widehat{R}^2\, d\theta \otimes d\theta + \widehat{R}^2\, \sin^2(\theta)\, d\phi \otimes d\phi.$$

In this section the Hodge map $\star$ will, throughout, refer to the metric of this background, approximately flat, spacetime domain. In the previous sections we have shown that the solutions $\Psi = \Psi_0$ and $\mathbb{A} = \Gamma\, e^1$ are exact solutions to the equations (4) and (6) in a particular LFRW metric. However these coupled non-linear *partial differential equations* on $\mathcal{M}$ have other solutions that may have relevance for the generation of local spatial inhomogeneities during the evolution of the Universe. If we assume that such solutions are perturbations of the exact cosmological solutions discussed above then a linearisation of (4) and (6) will offer a perturbative approach for finding them. Thus we analyse the first ϵ-order equations (37) and (38) for perturbative solutions about the zeroth ϵ-order solutions $\Psi^{(0)} = \Psi_0, \mathbb{A}^{(0)} = \Gamma\, e^1, F^{(0)} = 0$, i.e. the uncoupled *linear partial differential equations* for $\Psi^{(1)}$ and $\mathbb{A}^{(1)}$:

$$\star d \star d\Psi^{(1)} + \frac{W_1}{L_0^2}\, \Psi^{(1)} = 0 \tag{48}$$

$$\star d \star d\mathbb{A}^{(1)} + \frac{W}{L_0^2}\, \mathbb{A}^{(1)} = 0 \tag{49}$$

where

$$W_1 = \frac{1}{2}\zeta_0\, \Xi_1 \Gamma^2, \qquad W = \zeta_0\, \Xi_0.$$

Since the divergence operator $\delta \equiv \star d\star$ is nilpotent ($\delta^2 = 0$), all solutions of (49) satisfy

$$\delta \mathbb{A}^{(1)} = 0.$$

It is instructive to compare equations (48) and (49) with the classical Klein–Gordon and Proca field equations in Minkowski spacetime. These were introduced historically to accommodate short-range Yukawa-type static field (singular) solutions associated with *particles* having a real positive rest-mass m_0 and m_P respectively. Expressed in terms of their respective Compton wavelengths such equations take the form:

$$\star d \star d\Psi^{(1)} + \frac{m_0^2 c^2}{\hbar^2}\, \Psi^{(1)} = 0 \tag{50}$$

and

$$\star d \star \mathrm{d}\mathbb{A}^{(1)} + \frac{m_P^2 c^2}{\hbar^2}\,\mathbb{A}^{(1)} = 0$$

$$\delta\mathbb{A}^{(1)} = 0$$

(51)

where

$$m_0^2 = \frac{W_1\hbar^2}{L_0^2 c^2} \qquad \text{and} \qquad m_P^2 = \frac{W\hbar^2}{L_0^2 c^2}.$$

One sees that when the real parameters W and W_1 are strictly positive, (48) and (49) yield real values for m_0 and m_P but admit different types of solutions otherwise. Thus perturbative corrections to the cosmological solution $\mathbb{A}^{(0)}$ in the previous section require $W = -\mathcal{H}(0)^2\,\Delta^2 < 0$ with $\Delta \in \mathbb{R}$ so do not admit solutions with a real Proca rest-mass m_P. However the sign of ζ_0 is a-priori unconstrained and if W_1 is positive then perturbative corrections to the constant solution $\Psi^{(0)}$ can describe a perturbed scalar field associated with particles of mass m_0 that could, in the absence of further constraints, provide a dark-matter candidate since it has no direct coupling to the Maxwell field.

However the presence or absence of static solutions to either (48) or (49) does not imply the absence of dispersive propagating wave-packet (radiative) solutions constructed by superposition of separable mode solutions to these wave equations.

To illustrate this we consider an interesting class of exact solutions in local spatial cylindrical polar coordinates $\{\widehat{T}, \widehat{r}, \widehat{\phi}, \widehat{Z}\}$ in which the Minkowski metric tensor takes the form:

$$\widehat{g} = -c^2\,\mathrm{d}\widehat{T} \otimes \mathrm{d}\widehat{T} + \mathrm{d}\widehat{r} \otimes \mathrm{d}\widehat{r} + \widehat{r}^2\,\mathrm{d}\widehat{\phi} \otimes \mathrm{d}\widehat{\phi} + \mathrm{d}\widehat{Z} \otimes \mathrm{d}\widehat{Z}.$$

This choice of cylindrical polar coordinates will facilitate our discussion of particular exact solutions below. In these coordinates, a general integral representation of *complex solutions* for (48) may be constructed from a Fourier superposition of time harmonic and spatial cylindrical Helmholtz modes:

$$\Psi^{(w_1)}(\widehat{T}, \widehat{r}, \widehat{\phi}, \widehat{Z}) = \sum_{\text{modes}} \sum_{m=-\infty}^{\infty} \int_{s=0}^{\infty} \int_{\omega=-\infty}^{\infty} \widehat{\Psi}(\omega, s, m)\, J_m(\widehat{r}\,s)\, \mathrm{e}^{-\mathrm{i}\omega\widehat{T} + \mathrm{i}k(w_1,\omega,s)\widehat{Z} + \mathrm{i}m\widehat{\phi}}\, \mathrm{d}\omega\,\mathrm{d}s$$

in terms of the complex scalar Fourier amplitudes $\widehat{\Psi}$, the parameter $w_1 = W_1/L_0^2 \in \mathbb{R}$ and the complex branched dispersion relation:

$$k(w_1, \omega, s) = \sqrt{\frac{\omega^2}{c^2} - s^2 - w_1}\,.$$

(52)

Similarly a Fourier representation of solutions for (49) takes the form

$$\mathbb{A}^{(w)}(\widehat{T}, \widehat{r}, \widehat{\phi}, \widehat{Z}) = \sum_{\text{modes}} \sum_{m=-\infty}^{\infty} \int_{s=0}^{\infty} \int_{\omega=-\infty}^{\infty} \widehat{\mathbb{A}}(\omega, s, m)\, J_m(\widehat{r}\,s)\, \mathrm{e}^{-\mathrm{i}\omega\widehat{T} + \mathrm{i}k(w,\omega,s)\widehat{Z} + \mathrm{i}m\widehat{\phi}}\, \mathrm{d}\omega\,\mathrm{d}s$$

where

$$\widehat{\mathbb{A}}(\omega, s, m) = \widehat{\alpha}(\omega, s, m)\left(e^3 - \frac{c\,k(w,\omega,s)}{\omega}\,e^0\right)$$

in terms of $w = W/L_0^2 \in \mathbb{R}$, the complex scalar Fourier amplitudes $\widehat{\alpha}$, the 1-forms $e^0 = c\,\mathrm{d}\widehat{T}$, $e^3 = \mathrm{d}\widehat{Z}$ and the complex branched dispersion relation:

$$k(w, \omega, s) = \sqrt{\frac{\omega^2}{c^2} - s^2 - w}\,.$$

(53)

Domains for partitioning integral over $(\frac{\omega}{c}, s)$-plane for $w = 1$

Domains for partitioning integral over $(\frac{\omega}{c}, s)$-plane for $w = -1$
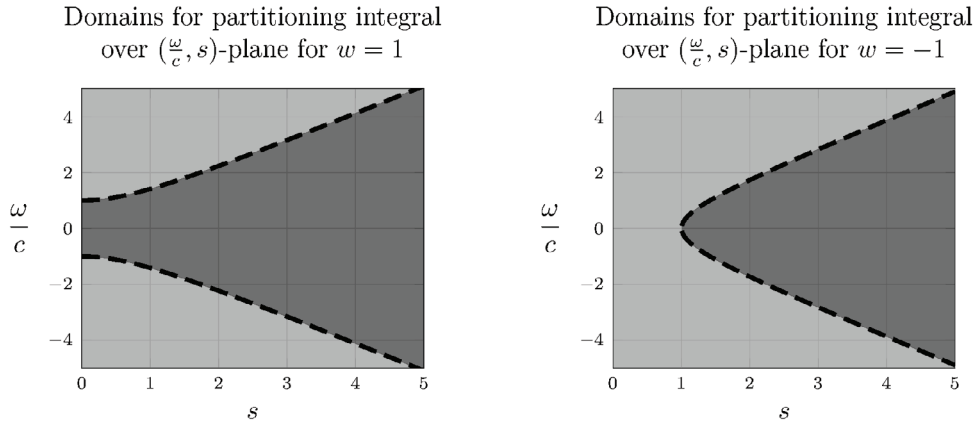
Figure 4. Light grey domains $(\omega^2/c^2 - s^2 - w > 0)$ distinguish contributions to the Fourier integral representations yielding progressive modes from dark grey domains $(\omega^2/c^2 - s^2 - w < 0)$ that contribute to evanescent modes.

The distinct mode types in the mode summations for $\Psi^{(w_1)}$ and $\mathbb{A}^{(w)}$ above arise from the distinct contributions to the integrations over ω and s for each integer m, determined by the sign of the argument of the square root function in the dispersion relations (52) and (53) respectively.

For each mode labelled by m in the summations for $\Psi^{(w_1)}$ and $\mathbb{A}^{(w)}$ above, the double integral over the half-plane $\omega \in [-\infty, \infty], s \in [0, \infty]$ can be partitioned into distinct contributions by branches of the hyperbolae $k(w_1, \omega, s) = 0$ and $k(w, \omega, s) = 0$ respectively. Contributions with k real yield progressive modes while those with k pure imaginary behave exponentially with respect to $\widehat{Z}$. The contributing physical modes are those that are spatially 'evanescent' for all $\widehat{Z}$. Figure 4 indicates how this partitioning depends on the sign of w_1 for the dispersion relation (52). A similar partition is determined by the branch points of the dispersion relation (53) where w has replaced w_1. In general the global behaviour of solutions generated from such mode summations depends critically on the structure of the scalar Fourier amplitudes $\widehat{\alpha}(\omega, s, m)$ and $\widehat{\Psi}(\omega, s, m)$.

8.1. Particular local fluctuations in Minkowski spacetime

The existence of particular examples of *axially symmetric* $(m = 0)$ dispersive complex wave-packet solutions of the equations (48) and (49) can be directly demonstrated using the methods of pre-potentials [28–30] and recently exploited to study the dynamic evolution of electromagnetic single-cycle laser pulses [31]. In terms of *strictly positive* arbitrary real constants Q_1, Q_2 with dimensions of length, introduce the dimensionless expression

$$\Lambda(\widehat{T}, \widehat{r}, \widehat{Z}) \equiv \frac{\sqrt{\widehat{r}^2 + [Q_1 + \mathrm{i}(\widehat{Z} - c\widehat{T})][Q_2 - \mathrm{i}(\widehat{Z} + c\widehat{T})]}}{L_0}. \tag{54}$$

Then one may verify directly that a particular complex axially-symmetric non-singular dispersive (radiative) pulse solution to (48) is given by the scalar fields:

$$\Psi_{W_1}^{(1)}(\widehat{T},\widehat{r},\widehat{Z}) = \frac{K_1\left(\sqrt{W_1}\,\Lambda(\widehat{T},\widehat{r},\widehat{Z})\right)}{\Lambda(\widehat{T},\widehat{r},\widehat{Z})} \qquad \text{when } W_1 > 0 \qquad (55)$$

$$\Psi_{W_1}^{(1)}(\widehat{T},\widehat{r},\widehat{Z}) = \frac{\mathcal{C}\left(\sqrt{-W_1}\,\Lambda(\widehat{T},\widehat{r},\widehat{Z})\right)}{\Lambda(\widehat{T},\widehat{r},\widehat{Z})} \qquad \text{when } W_1 < 0 \qquad (56)$$

where K_1 is a modified second kind Bessel function and $\mathcal{C}$ is either the Hankel function $H_1^{(1)}$ or $H_1^{(2)}$.

It has been shown in [32, 33] how solutions to certain linear scalar wave equations can be used to generate solutions to certain linear vector wave equations. A similar process is available here to generate a particular solution to (49) from the solutions (55) or (56) with W_1 replaced by W. Thus for *any real* $W \neq 0$, complex dispersive ν-chiral vector wave-packet solutions to (49) are given in terms of (55) or (56) by

$$\mathbb{A}_\nu^{(1)}(\widehat{T},\widehat{r},\widehat{Z}) = \star d\left(\Psi_W^{(1)}(\widehat{T},\widehat{r},\widehat{Z})\,\Pi_\nu\right), \qquad \nu = \pm 1, 0 \qquad (57)$$

where the closed ($d\Pi_\nu = 0$) and co-closed ($\delta\Pi_\nu = 0$) complex chiral 2-forms Π_ν are given by

$$\Pi_{\pm 1} = d(\widehat{r}e^{\pm i\widehat{\phi}}) \wedge c\,d\widehat{T}, \qquad \Pi_0 = d\widehat{Z} \wedge c\,d\widehat{T}.$$

The detailed 'dispersive' behaviour of these axially-symmetric complex wave-packets, as functions of $(\widehat{T},\widehat{r},\widehat{Z})$, is determined by the ratio Q_1/Q_2 and the parameters W,W_1. The real (or imaginary) parts describe real non-singular solutions to (48) and (49). By construction the expression (54) generates solutions with turning points at particular values when $\widehat{T} = 0$. The location of these turning points can easily be changed by shifting the origin of the spatial coordinates. Similarly the direction of the $\widehat{Z}$-axis of symmetry can be arbitrarily rotated to another direction [34]. Those solutions involving the K_1 and $H_1^{(2)}$ Bessel functions exhibit bounded axially-symmetric wave-packet profiles that spread in $\widehat{r}$ and $\widehat{Z}$ as $\widehat{T}$ increases. By contrast those involving $H_1^{(1)}$ Bessel functions exhibit growing amplitudes as $\widehat{T}$ increases and are unphysical solutions.

Given the properties of the particular scalar and vector perturbations discussed in this section, one can summarise the following criteria on the basic parameters for the perturbed cosmological model. We demand $W \equiv \zeta_0 \Xi_0 < 0$ to be consistent with $q(0) < 0$ and $\Xi_1 \equiv \mathcal{U}''(\Psi^{(0)}) > 0$ for a potential minimum at $\Psi^{(0)}$. However solutions that admit Klein–Gordon and Proca positive rest-mass solutions require $W_1 > 0$ and $W > 0$ respectively where $W_1 \equiv \zeta_0 \Xi_1 \Gamma^2/2$ with Γ real. Furthermore solutions with arbitrary non-zero values of W_1 and W admit scalar field and vector field physical wave-packet profiles respectively. Since ζ_0 and $\Xi_0 \equiv \mathcal{U}(\Psi^{(0)})$ remain unconstrained we exhibit these conditions as designated domains in the $\Xi_0 - \zeta_0$ plane in figure 5. Shaded domains in these figures:

  (i) admit construction of a viable cosmological model from $S(t)$ with one real root.

 (ii) admit scalar perturbations $\Psi^{(1)}$ that satisfies the Klein–Gordon equation (50) with a real rest-mass m_0.

(iii) admit vector perturbations $\mathbb{A}^{(1)}$ satisfying the Proca equation (51) with a real rest-mass m_P.

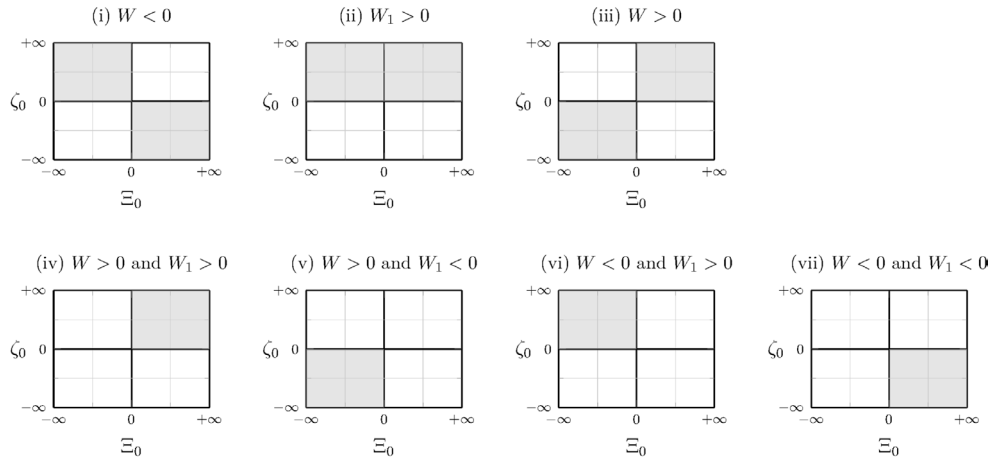(iv) admit both (ii) and (iii).

Figure 5. $\{W, W_1, \zeta_0, \Xi_0\}$ correlations for solutions for the Proca and Klein–Gordon wave equations.

(v) admit vector perturbations $\mathbb{A}^{(1)}$ satisfying a Proca equation with a real rest-mass m_P and spatially bounded dispersive wave packets (55).

(vi) admit vector perturbations $\mathbb{A}^{(1)}$ described by spatially bounded dispersive wave packet solutions (57) and scalar perturbations $\Psi^{(1)}$ described by solutions with a real rest-mass m_0 and spatially bounded dispersive wave packet solutions (55).

(vii) admit vector perturbations $\mathbb{A}^{(1)}$ described by solutions with spatially bounded dispersive wave packets (57) and scalar perturbations $\Psi^{(1)}$ described by solutions with spatially bounded dispersive wave packets (55).

These designated domains indicate that there exist real values of the couplings ζ_0 and Ξ_0 permitting the construction of the cosmological solution with $W < 0$ discussed in this paper. Such couplings then permit the construction of perturbed solutions $\Psi^{(1)}$ and $\mathbb{A}^{(1)}$ to the linear wave equations (48) and (49) in a background domain that is approximately Minkowskian. The former has particular solutions characteristics of a Klein–Gordon solution with a real particle mass m_0 or an axially symmetric dispersive propagating wave packet while the latter has only axially symmetric dispersive propagating polarised vector wave packet solutions. Assuming that the history of such dispersive fluctuations are currently detectable, an observational signature may reside in the influence of a primordial vector field polarisation on visible matter (or electromagnetic fields) or effects due to a primordial massive Klein–Gordon field.

9. Conclusions

In this article we have discussed a new paradigm for exploring a number of puzzling aspects in modern cosmology and their implications for astrophysics and observable astronomy. In this paradigm we argue that an evolutionary description of the Universe is best formulated in terms of a series of successive approximations based on a viable *cosmography* derived from current observations with a minimum number of phenomenological constraints on the dynamics of the unobservable early Universe. Consequently assumptions about the states of matter during such epochs in our paradigm are replaced by a series of retro-dictions from a coupled system of field equations with initial conditions based on current data. We depart from many standard

cosmological models, with their use of different isotropic fluid models in different epochs, by using a single anisotropic material fluid model in an Einstein–Maxwell-vector-scalar-fluid system of master equations. The Maxwell, vector and scalar fields are coupled to gravity and themselves in such a way that an exact analytic approach to a cosmological solution for the spacetime metric, vector, scalar and Maxwell fields can be found without the need to impose any a-priori equation of state for the material fluid. Instead, its equation of state in the cosmological sector is induced from the Einstein equation containing a stress–energy–momentum tensor without a cosmological constant. By establishing all solutions on a spacetime with the topology $\mathcal{I} \times S^3, \mathcal{I} \in \mathbb{R}$ the spacetime metric falls within the LFRW class of metrics possessing spatial sections with topology S^3 and the associated scale factor describes five distinct spacetime geometries. Motivated by the recent estimates of the negative de-acceleration parameter we use the current value of the Hubble parameter to select a viable cosmology with a single singular state and an exponentially expanding scale factor. Its predicted history leads to the value 13.2×10^9 years for the age of the Universe and a predicted value of 1.48 for the (unmeasured) 'jerk' parameter. Based on this history we have verified that, over an interval from the big-bang, including the current epoch, the dominant-energy, weak-energy and null-energy conditions are satisfied and only the strong-energy conditions are (weakly) violated.

By neglecting any back-reaction of the fields on the LFRW spacetime we have derived a series of coupled linear PDE's that determine the vector, scalar and Maxwell field fluctuations by the method of successive approximations. In this scheme one finds that, to lowest *and* first order, the primordial vector field remains 'dark' (i.e. has no direct coupling to the Maxwell field at those orders). Furthermore we show that at second order the Maxwell field acquires an electric current source induced from lower order scalar fields and source-free Maxwell perturbations. We argue that these currents offer a potential mechanism for initiating a thermalisation process between matter and electromagnetic radiation. Based on the COBE observation of a microwave dominant Planck spectrum of cosmological origin and a choice of 4000 K for the temperature at radiation decoupling from matter, our cosmological model history predicts a time interval of 8.23×10^3 years between the big-bang and the decoupling era.

By restricting to spacetime domains where the local effects of gravity may be neglected we use the parameters fixed by cosmography and the coupling constants that enter into the master field equations to explore the relation of the vector and scalar perturbations to the solutions of the historic Proca and Klein–Gordon equations in Minkowski spacetime. We show that there exist couplings where both vector and scalar dispersive wave-packet solutions arise and where the scalar perturbation may give rise to Yukawa solutions associated with a massive Klein–Gordon particle with rest mass given by

$$m_0^2 = \frac{\zeta_0 \Xi_1 \Gamma^2 \hbar^2}{2L_0^2 c^2} \qquad (\zeta_0 > 0).$$

In S.I. units ($L_0 = 1$), this has the value

$$m_0^2 = 0.616 \times 10^{-84} \zeta_0 \Xi_1 \Gamma^2.$$

Dark matter searches are then circumscribed by the values of the dimensionless parameters ζ_0, Ξ_0 and Γ which must be determined independently from other astrophysical observations. These include applications of the perturbation scheme in section 8 to gravitational lensing by galaxies, bounds on primordial magnetic fields, CMBR anisotropies and dark-energy constraints.

Aside from such local perturbative features the global aspects of our model give rise to a number of novel electromagnetic solutions that owe their existence to the three-sphere

topology of space. The symmetry of the electric and magnetic bi-pole solutions discussed in section 4 may have relevance to charge conjugation and parity inversion symmetry in the early Universe while primordial extragalactic magnetic fields may owe their existence to evolving 'Hopfion-like' solutions with magnetic helicity.

Further investigations based on the Einstein-vector-scalar-Maxwell-fluid paradigm discussed above may have implications for other more challenging problems in cosmology and astrophysics and will be discussed elsewhere. However, it is important to recognise that to confront our model with such astrophysical phenomena, the latest experimental data [27] needs to be analysed beyond the standard paradigm.

Acknowledgments

Appendix

In this appendix the stereographic mapping employed in section 3 in the construction of a spacetime LFRW metric tensor, in terms of Maurer–Cartan forms on S^3, is described.

An $SO(3)$ group element U can be expressed in terms of the Pauli matrices $\{\sigma_1, \sigma_2, \sigma_3\}$ and group coordinates $\{\alpha_1, \alpha_2, \alpha_3\}$ as

$$U = \exp\left[i(\alpha_1\sigma^1 + \alpha_2\sigma^2 + \alpha_3\sigma^3)\right].$$

The three pure imaginary Maurer–Cartan 1-forms

$$\widetilde{\Theta}^k = \frac{1}{2}\mathrm{tr}(\sigma^k \, U^{-1}\mathrm{d}U) \qquad (k = 1, 2, 3)$$

in the real coordinate cobasis $\{\mathrm{d}\alpha_1, \mathrm{d}\alpha_2, \mathrm{d}\alpha_2\}$ are then:

$$\widetilde{\Theta}^k = \frac{i}{\alpha^3}\sum_{j=1}^{3}\widetilde{\Theta}^k_j\,\mathrm{d}\alpha_j$$

where

$$\widetilde{\Theta}^1_1 = (\alpha_2^2 + \alpha_3^2)\cos(\alpha)\sin(\alpha) + \alpha_1^2\,\alpha$$
$$\widetilde{\Theta}^1_2 = \alpha_1\,\alpha_2\,(\alpha - \cos(\alpha)\sin(\alpha)) - \alpha_3\,\alpha\sin^2(\alpha)$$
$$\widetilde{\Theta}^1_3 = \alpha_1\,\alpha_3\,(\alpha - \cos(\alpha)\sin(\alpha)) - \alpha_2\,\alpha\sin^2(\alpha)$$

$$\widetilde{\Theta}^2_1 = \alpha_1\,\alpha_2\,(\alpha - \cos(\alpha)\sin(\alpha)) + \alpha_3\,\alpha\sin^2(\alpha)$$
$$\widetilde{\Theta}^2_2 = (\alpha_1^2 + \alpha_3^2)\cos(\alpha)\sin(\alpha) + \alpha_2^2\,\alpha$$
$$\widetilde{\Theta}^2_3 = \alpha_2\,\alpha_3\,(\alpha - \cos(\alpha)\sin(\alpha)) - \alpha_1\,\alpha\sin^2(\alpha)$$

$$\widetilde{\Theta}_1^3 = \alpha_1\,\alpha_3\,(\,\alpha - \cos(\alpha)\sin(\alpha)\,) - \alpha_2\,\alpha\,\sin^2(\alpha)$$

$$\widetilde{\Theta}_2^3 = \alpha_2\,\alpha_3\,(\,\alpha - \cos(\alpha)\sin(\alpha)\,) + \alpha_1\,\alpha\,\sin^2(\alpha)$$

$$\widetilde{\Theta}_3^3 = (\alpha_1^2 + \alpha_2^2)\cos(\alpha)\sin(\alpha) + \alpha_3^2\,\alpha$$

and

$$\alpha^2 \equiv \alpha_1^2 + \alpha_2^2 + \alpha_3^2.$$

In these equations the coordinates are defined in the intervals $-\pi < \alpha_j < \pi$ and $0 \leqslant \alpha < \pi$. To facilitate the construction of the LFRW metric from the Maurer–Cartan forms $\widetilde{\Theta}^k$ on S^3 we introduce the stereographic coordinate transformation $\{\alpha_1, \alpha_2, \alpha_3\} \mapsto \{\xi_1, \xi_2, \xi_3\}$:

$$\alpha_1 = \left(\frac{\xi^2 - 1}{\xi^2 + 1}\right)\chi, \quad \alpha_2 = \left(\frac{2\xi_3}{\xi^2 + 1}\right)\chi, \quad \alpha_3 = \left(\frac{2\xi_2}{\xi^2 + 1}\right)\chi \qquad \text{(A.1)}$$

where

$$\chi \equiv \frac{\arctan\left(\sqrt{1 - \omega^2}, \omega\right)}{\sqrt{1 - \omega^2}},$$

and

$$\xi^2 \equiv \xi_1^2 + \xi_2^2 + \xi_3^2, \qquad \omega \equiv \frac{2\,\xi_1}{\xi^2 + 1}.$$

The two-argument function $\arctan(y, x)$ computes the principal value of the argument of the complex number $x + iy$ such that $\arctan(y, x) \in (-\pi, \pi]$. In the $\{\xi_1, \xi_2, \xi_3\}$ chart the Maurer–Cartan 1-forms $\{\widetilde{\Theta}^1, \widetilde{\Theta}^2, \widetilde{\Theta}^3\}$ become:

$$\Theta^1 = \frac{4i}{(1 + \xi_1^2 + \xi_2^2 + \xi_3^2)^2}\left(\frac{1}{2}(1 + \xi_1^2 - \xi_2^2 - \xi_3^2)d\xi_1 + (\xi_1\xi_2 + \xi_3)d\xi_2 + (\xi_1\xi_3 - \xi_2)d\xi_3\right)$$

$$\Theta^2 = \frac{4i}{(1 + \xi_1^2 + \xi_2^2 + \xi_3^2)^2}\left((\xi_1\xi_2 - \xi_3)d\xi_1 + \frac{1}{2}(1 + \xi_2^2 - \xi_1^2 - \xi_3^2)d\xi_2 + (\xi_2\xi_3 + \xi_1)d\xi_3\right)$$

$$\Theta^3 = -\frac{4i}{(1 + \xi_1^2 + \xi_2^2 + \xi_3^2)^2}\left((\xi_1\xi_3 + \xi_2)d\xi_1 + (\xi_2\xi_3 - \xi_1)d\xi_2 + \frac{1}{2}(1 + \xi_3^2 - \xi_1^2 - \xi_2^2)d\xi_2\right).$$

In these equations the coordinates are defined in the intervals $-\infty < \xi_j < \infty$ and cover S^3 minus the point where $\xi^2 \to \infty$ or, under the transformation:

$$\left\{\xi_1 = \frac{1}{2}r\sin(\theta)\cos(\phi), \quad \xi_2 = \frac{1}{2}r\sin(\theta)\sin(\phi), \quad \xi_3 = \frac{1}{2}r\cos(\theta)\right\},$$

at the point where $r^2 \to \infty$.

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