

Classical Dynamics of Free Electromagnetic Laser Pulses

S. Goto^a, R. W. Tucker^b, T. J. Walton^c

^a*Institute for Molecular Science, 38 Nishigo-Naka, Myodaiji, Okazaki 444-8585, Japan*

^b*Department of Physics, University of Lancaster and Cockcroft Institute, Daresbury Laboratory, Warrington, UK*

^c*Department of Mathematics, University of Bolton, Deane Campus, Bolton, UK*

Abstract

We discuss a class of exact finite energy solutions to the vacuum source-free Maxwell field equations as models for multi- and single cycle laser pulses in classical interaction with relativistic charged test particles. These solutions are classified in terms of their *chiral* content based on their influence on particular charge configurations in space. Such solutions offer a computationally efficient parameterization of compact laser pulses used in laser-matter simulations and provide a potential means for experimentally bounding the fundamental length scale in the generalized electrodynamics of Bopp, Landé and Podolsky.

Keywords: Laser pulse, finite energy, Maxwell equations, Bopp-Landé-Podolsky

Advances in laser technology have made possible the exploration of physical processes on unprecedented temporal and spatial scales. They have also opened up new possibilities for accelerating charged particles using laser-matter interactions. Multi- and single cycle high intensity ($10^{10} - 10^{15}$ Watts/cm²) laser pulses can be produced using Q-switching or mode-locking techniques [1]. Such pulses can accelerate charged particles such as electrons to relativistic speeds where radiation reaction and quantum effects may influence their dynamics. Lower intensity pulses have also been used as diagnostic tools for exploring the structure of plasmas in various states [2, 3]. In order to interpret experimental data involving classical laser interactions with both charged and neutral matter, theoretical models [4–7] rely crucially on parameterizations of the electromagnetic fields in laser pulses, particularly in situations where traditional formulations using monochromatic or paraxial-beam approximations have limitations [8–10].

In this Letter we discuss a viable methodology for parameterizing a particular class of propagating solutions to the source free *classical* Maxwell equations in vacuo that offers an efficient means to explore the classical effects of compact laser pulses on free electrons in dynamical regimes where quantum effects are absent. The parameterization is based on a remarkable class of explicit solutions of the scalar wave equation found by Ziolkowski [11–15] following pioneering work by Brittingham [16]. Such solutions can be used to construct classical Maxwell solutions with bounded total electromagnetic energy and fields bounded in all three spatial directions. With simple analytic structures their diffractive properties can be readily determined

together with the behaviour of charged particle-pulse interactions over a broad parameter range without recourse to expensive numerical computation. Finally, we argue that such parameterizations can be used to find compact finite energy solutions to other linear wave equations. This is illustrated by showing that the generalized theory of Bopp [17], Landé [18] and Podolsky [19] admits such particular solutions that reduce to the Maxwell solutions when a fundamental length parameter in their theory tends to zero. Compact laser pulses in this theory might be used to explore properties of the theory by searching experimentally for bounds on this parameter.

If a complex scalar field α satisfies $\square\alpha = 0$ on spacetime and $\Pi_{\mu\nu}$ is any covariantly constant (degree 2) anti-symmetric tensor field on spacetime (i.e. $\Pi_{\mu\nu;\delta} = 0$) for all $\mu, \nu, \delta = 0, 1, 2, 3$, then the complex tensor field $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ satisfies the source free Maxwell equations in vacuo with:

$$A_\nu = \partial_\gamma (\alpha \Pi_{\mu\beta}) \epsilon^{\gamma\mu\beta}{}_\nu \sqrt{|g|} \quad (1)$$

where $|g|$ is the determinant of the spacetime metric and $\epsilon^{\gamma\mu\beta}{}_\nu$ denotes the Levi-Civita alternating symbol. In the following g refers to the Minkowski metric tensor field, in which case the components $\Pi_{\mu\nu}$ can be used to encode three independent Hertz vector fields and their duals¹.

General solutions to $\square\alpha = 0$ can be constructed by Fourier analysis. In cylindrical polar Minkowski coordinates $\{t, r, z, \theta\}$, axially symmetric solutions propagating along the z -axis have, for $z \geq 0$, the double integral repre-

¹In the language of differential forms on Minkowski spacetime $A = \star d(\alpha\Pi)$, $F = dA$ where $d \star d\alpha = 0$, the 2-form Π satisfies $\nabla\Pi = 0$ and \star denotes the Hodge map associated with g .

Email addresses: sgoto@ims.ac.jp (S. Goto),
r.tucker@lancaster.ac.uk (R. W. Tucker),
t.walton@bolton.ac.uk (T. J. Walton)

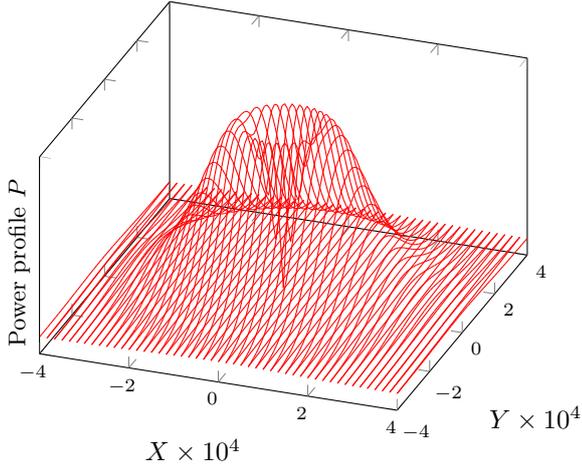


Figure 1: Power profile P of the (CM, 1) laser pulse at $Z = 0$, $T = 0$ with parameters $\{\Lambda = 600, \Psi_1 = 1, \Psi_2 = 1000, \Phi = 0.001, \Xi = 1\}$

sensation $\alpha(t, r, z) = \int_{-\infty}^{\infty} d\omega e^{-i\omega t} \tilde{\alpha}(\omega, r, z)$ where:

$$\begin{aligned} \tilde{\alpha}(\omega, r, z) = & \int_0^{\frac{\omega}{c}} k f_{\omega}(k) J_0(kr) e^{\pm iz\sqrt{(\frac{\omega}{c})^2 - k^2}} dk \\ & + \int_{\frac{\omega}{c}}^{\infty} k f_{\omega}(k) J_0(kr) e^{-z\sqrt{k^2 - (\frac{\omega}{c})^2}} dk \end{aligned}$$

in terms of the zero order Bessel function and the speed of light in vacuo c .

Conditions on the Fourier amplitudes $f_{\omega}(k)$ can be given so that the Hertz procedure above gives rise to real singularity free electromagnetic fields with finite total electromagnetic energy. A particularly simple class of pulses that can be generated in this way follows from the complex axi-symmetric scalar solution:

$$\alpha(t, r, z) = \frac{\ell_0^2}{r^2 + (\psi_1 + i(z - ct))(\psi_2 - i(z + ct))} \quad (2)$$

where ℓ_0, ψ_1, ψ_2 are arbitrary real positive definite parameters with physical dimensions of length. The relative sizes of ψ_1 and ψ_2 determine both the direction of propagation along the z -axis of the dominant maximum of the pulse profile and the number of spatial cycles in its peak magnitude. When $\psi_1 \gg \psi_2$, the dominant maximum propagates along the z -axis to the right. The parameter ℓ_0 determines the magnitude of such a maximum. The structure of such solutions has been extensively studied in [20, 21] in conjunction with particular choices of $\Pi_{\mu\nu}$ together with generalizations discussed in [22, 23].

In general the six anti-symmetric tensors with components $\delta_{[\gamma}^{\mu} \delta_{\sigma]}^{\nu}$ in a Minkowski Cartesian coordinate system are covariantly constant and can be used to construct a complex eigen-basis of antisymmetric *chiral* tensors $\Pi^{s, \kappa}$, with $s \in \{\text{CE, CM}\}$ and $\kappa \in \{-1, 0, 1\}$, satisfying

$$\mathcal{O}_z \Pi^{s, \kappa} = \kappa \Pi^{s, \kappa} \quad (3)$$

where the operator \mathcal{O}_z represents θ rotations about the z -axis generated by $-i\partial_{\theta}$ on tensors². These in turn can be used to construct a *complex basis* of chiral eigen-Maxwell tensor fields $F^{s, \kappa}$. The index s indicates that the CE (CM) chiral family contain electric (magnetic) fields that are orthogonal to the z -axis when $\kappa = 0$. The chiral eigen-fields $F^{s, 0}$ inherit the axial symmetry of $\alpha(t, r, z)$ while those with $\kappa = \pm 1$ do not. The directions of electric and magnetic fields in any of these Maxwell solutions depend on their location in the pulse and the concept of a pulse polarisation is not strictly applicable. The chiral content as defined here can be used in its place. Non-chiral pulse configurations can be constructed by superposition $\sum_s \sum_{\kappa} F^{s, \kappa} \mathcal{C}^{s, \kappa}$ with arbitrary complex coefficients $\mathcal{C}^{s, \kappa}$.

The energy, linear and angular momentum of the pulse in vacuo can be calculated from the components $T_{\mu\nu}$ of the Maxwell stress-energy tensor $T_{\mu\nu} = -\frac{1}{4}g_{\mu\nu}\mathcal{F}^{\alpha\beta}\mathcal{F}_{\alpha\beta} - \mathcal{F}_{\mu\alpha}\mathcal{F}^{\alpha\nu}$ where $\mathcal{F}_{\mu\nu} = \text{Re}(F_{\mu\nu})$. If \mathbf{e} and \mathbf{b} denote time-dependent *real* electric and magnetic 3-vector fields associated with any pulse solution, its total electromagnetic energy \mathcal{J} , for a fixed set of parameters and any z , is calculated from

$$\mathcal{J} = \frac{1}{\mu_0} \int_{-\infty}^{\infty} dt \int_S (\mathbf{e} \times \mathbf{b}) \cdot d\mathbf{S} \quad (4)$$

where S can be any plane with constant $z = z_0 > 0$. For *spatially compact* pulse fields in vacuo this coincides with the total pulse electromagnetic energy

$$\mathcal{E} = \int_{\mathcal{V}} \rho d\mathcal{V} = \int_{-\infty}^{\infty} dz \int_0^{2\pi} d\theta \int_0^{\infty} r dr \rho(t, r, z, \theta) \quad (5)$$

where $\rho \equiv \frac{1}{2} (\epsilon_0 \mathbf{e} \cdot \mathbf{e} + \frac{\mathbf{b} \cdot \mathbf{b}}{\mu_0})$ is integrated over all space \mathcal{V} . This follows since $\nabla \cdot (\mathbf{e} \times \mathbf{b}) = -\mu_0 \partial_t \rho$. To correlate \mathcal{J} with other laser pulse properties and the choice of parameters, we bring the pulse into classical interaction with one or more charged point particles. The world-line of a single particle, parameterized in arbitrary coordinate as $x^{\mu} = \xi^{\mu}(\tau)$ with a parameter τ , is taken as a solution of the coupled non-linear differential equations

$$\mathcal{A}_{\mu}(\tau) = \frac{q}{m_0 c^2} \mathcal{F}_{\mu\nu}(\xi(\tau)) V^{\nu}(\tau) \quad (6)$$

in terms of the particle charge q and rest mass m_0 , for some initial conditions $\xi(0), V(0)$, where the particle 4-velocity satisfies $V^{\nu} V_{\nu} = -1$ and its 4-acceleration is expressed in terms of the Christoffel symbols $\Gamma^{\delta\beta}_{\mu}$ as $\mathcal{A}_{\mu} = \partial_{\tau} V_{\mu}(\tau) + V_{\delta}(\tau) V_{\beta}(\tau) \Gamma^{\delta\beta}_{\mu}(\xi(\tau))$. In the following, radiation reaction and inter-particle forces are assumed negligible. From the solution $\xi(\tau)$ one can determine the increase (or decrease) in the relativistic kinetic energy transferred from the electromagnetic pulse to any particle and the nature of its trajectory in the laboratory frame. This

²In terms of the Lie derivative, $\mathcal{O}_z = -i\mathcal{L}_{\partial_{\theta}}$ and $\Pi^{\text{CE}, \pm 1} = d(x \pm iy) \wedge dt$, $\Pi^{\text{CE}, 0} = dz \wedge dt$, $\Pi^{\text{CM}, \kappa} = \star \Pi^{\text{CE}, \kappa}$ where $x = r \cos(\theta)$, $y = r \sin(\theta)$

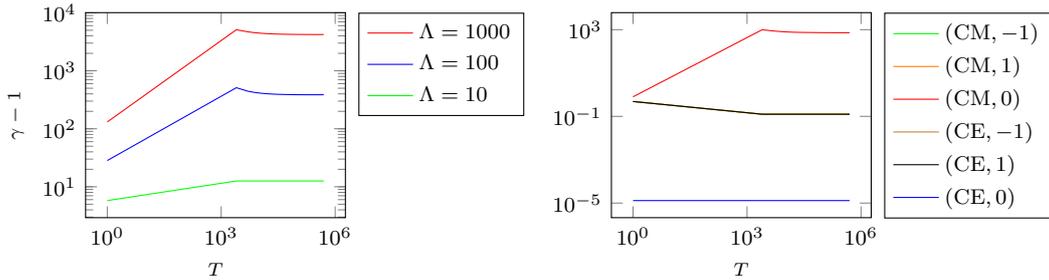


Figure 3: Specific kinetic energy transfer to any charged test particle. On the left, the (CM,1) pulse has parameters $\{\Psi_1 = 1, \Psi_2 = 1000, \Phi = 0.001, \Xi = 1\}$. On the right, relative values of $\gamma - 1$ are displayed for various (s, κ) pulses with parameters $\{\Lambda = 1, \Psi_1 = 1, \Psi_2 = 1000, \Phi = 0.001, \Xi = 1\}$. The differences between the energy transfers for some pulses appear indistinguishable relative to others owing to the logarithmic scales employed. In all cases, the charged particle has initial position $\{R(0) = 1, \theta(0) = \frac{\pi}{2}, Z(0) = 1\}$ and initial velocity $\{\dot{R}(0) = 0, \dot{Z}(0) = \frac{1}{200}, \dot{\theta}(0) = 0\}$.

We deduce that the momentum and angular momentum [24] in the propagation direction can transfer an impulsive force and torque respectively to charges lying in an orthogonal plane. More generally, the classical configurations of a high energy pulse labelled CE and CM can be distinguished by their interaction with different arrangements of charged matter.

Furthermore, by a suitable choice of parameters, (CE, κ) type modes can be constructed that yield the same physical properties $(\mathcal{J}, z_{rg}, z_w, \beta)$ for all κ . Similarly the (CM, κ) type modes yield a κ independent set with physical properties distinct from those determined by the (CE, κ) modes. The pulse group speed magnitudes (as defined above) of all these configurations are determined numerically and are bounded above by the value c . To illustrate some of these statements, table 1 summarizes the MKS laser pulse characteristics for a specific choice of $\{\Psi_1, \Psi_2, \Phi, \Xi\}$ and various values of Λ .

Motivated by the desire to ameliorate the divergences in perturbative QED a number of generalized theories of electromagnetism have been proposed. To date, there is little experimental evidence for testing their predicted departures from Maxwell's theory. However, with the increase in laser technology one may now be entering regimes that may discriminate between such theories. In particular, Bopp-Landé-Podolsky electrodynamics is linear in the

s	CE	CE	CM	CM	CM
κ	0	± 1	0	± 1	± 1
\mathcal{E} (J)	8.07	7.69	8.41	7.69	76903
Λ	40000	1	0.05	1	100
z_{rg} (m)	1.44	2.43	0.899	1.80	1.80
t_0 (ps)	3	3	2	3	3
z_w (mm)	0.899	0.899	0.600	0.899	0.899
$r_s(0)$ (m)	0.018	0.009	0.012	0.009	0.009
I (TW cm $^{-2}$)	2.646	1.009	0.930	1.009	10090

Table 1: Table showing MKS laser characteristics for various (s, κ) laser pulse configurations with parameters $\{\Psi_1 = 1, \Psi_2 = 1000, \Phi = 0.001, \Xi = 1\}$.

electromagnetic fields, contains a fundamental length λ_0 and approaches Maxwell's theory when this length tends to zero. Unlike Maxwell's theory, it contains solutions describing static point charges with *finite* electromagnetic energy.

The classical source-free Bopp-Landé-Podolsky field equations in vacuo for the complex electromagnetic field tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ are:

$$\partial^\mu F_{\mu\nu} - \lambda_0^2 \square \partial^\mu F_{\mu\nu} = 0. \quad (8)$$

Clearly, all classical source-free vacuum Maxwell solutions satisfy these equations. However, there exist additional solutions to (8) that are *not* source-free vacuum Maxwell solutions. For example, such additional solutions of (8) follow from (1) provided:

$$\square \alpha + \lambda_0^2 \square^2 \alpha = 0. \quad (9)$$

Furthermore, it follows that as $\lambda_0 \rightarrow 0$, one recovers the class of Maxwell solutions discussed above. For non-zero λ_0 , there exist particular non-Maxwellian solutions to (9) satisfying

$$\square \alpha + \frac{1}{\lambda_0^2} \alpha = 0.$$

In Minkowski cylindrical coordinates $\{t, r, \theta, z\}$, a non-separable axi-symmetric compact laser solution is given for arbitrary real Λ_0 by:

$$\alpha(t, r, z) = \frac{\Lambda_0^2 K_1 \left(\frac{\zeta(r, t, z)}{\lambda_0} \right)}{\zeta(t, r, z)}$$

where

$$\zeta(t, r, z) \equiv \sqrt{r^2 + [\psi_1 + i(z - ct)] [\psi_2 - i(z + ct)]}$$

in terms of arbitrary real positive-definite parameters ψ_1, ψ_2 that determine propagation along the positive z -axis and where $K_1(z)$ denotes the first order modified Bessel function of the second kind. The function $\zeta(t, r, z)$ is positive for all real r, t, z and hence both the real and imaginary

parts of $\alpha(t, r, z)$ are bounded.

The interaction of the Bopp-Landé-Podolsky pulse with classical charged test particles follows from the divergence of the Bopp-Landé-Podolsky stress-energy-momentum tensor [25]. This in turn leads to the same equation of motion (6) for the test particle trajectories [25] and may offer an experimental means to discriminate between the Maxwell and Bopp-Landé-Podolsky descriptions of source-free electromagnetic fields in vacuo.

Further details of this approach using scalar field modulated Hertz potentials based on the construction (1) for finding spatially compact solutions to other linear tensor and spinor wave equations will be presented elsewhere.

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