J. Group Theory, Ahead of Print
DOI 10.1515/jgth-2021-0040

Completely bounded homomorphisms of the Fourier algebra revisited

Matthew Daws*

Communicated by Adrian Ioana

Abstract. Assume that $A(G)$ and $B(H)$ are the Fourier and Fourier–Stieltjes algebras of locally compact groups G and H, respectively. Ilie and Spronk have shown that continuous piecewise affine maps $\alpha\colon Y \subseteq H \to G$ induce completely bounded homomorphisms $\Phi\colon A(G) \to B(H)$ and that, when G is amenable, every completely bounded homomorphism arises in this way. This generalised work of Cohen in the abelian setting. We believe that there is a gap in a key lemma of the existing argument, which we do not see how to repair. We present here a different strategy to show the result, which instead of using topological arguments, is more combinatorial and makes use of measure-theoretic ideas, following more closely the original ideas of Cohen.

1 Introduction

Cohen [1] classified all bounded homomorphisms from the group algebra $L^1(G)$ to the measure algebra $M(H)$, for locally compact abelian groups G, H; this was later expounded with different proofs by Rudin [8]. The characterisation given was in terms of Pontryagin duals and so, in modern language, is more naturally stated as studying homomorphisms between the Fourier algebra $A(\widehat{G})$ and the Fourier–Stieltjes algebra $B(\widehat{H})$, these algebras being introduced by Eymard [2] for arbitrary locally compact groups. It is now widely recognised that it is natural to work in the category of operator spaces and completely bounded maps when studying Fourier algebras of non-abelian groups. In [3], Ilie provided a generalisation of Cohen's result for discrete groups, characterising completely bounded homomorphisms $A(G) \to B(H)$ in terms of coset rings of H, and piecewise affine maps. In [4], Ilie and Spronk extended this result to all locally compact groups, making use of open coset rings. We also mention [7] which shows similar results for merely contractive (not completely bounded) homomorphisms.

We do not fully follow the proof given in [4], nor that of Rudin in [8]. The proof of a key lemma in [4] appears to have a gap, and we have been unable to see how to repair this. In this paper, we return to Cohen's original proof for

The author is partially supported by EPSRC grant EP/T030992/1.

inspiration and provide a new proof of the main result of [4], using a more complicated combinatorial argument and using measure theory ideas. Cohen's proof is, in places, firmly a proof about abelian groups, which necessitates new ideas in the non-abelian setting.

The papers [3, 4] have been widely cited, used and generalised in the 15 years since they were published, and we feel it is important that this result has a solid, careful proof attached to it. Let us now provide some further background, which will allow us to be precise about the perceived problems in [4,8]. We will then give an overview of our alternative strategy. The rest of the paper is concerned with the precise details of executing this new strategy.

Let X be a set. For us, a *ring of subsets* of X, say $\mathcal{S}$, is a (non-empty) collection of subsets of X such that, for $A, B \in \mathcal{S}$, also $A \cap B, A \cup B \in \mathcal{S}$, and if $A \in \mathcal{S}$, also $X \setminus A \in \mathcal{S}$. Then $\emptyset$, and so also X, are in $\mathcal{S}$, and notice that $\mathcal{S}$ is also closed under taking symmetric differences.

Let G be a group, and let $H \leq G$ be a subgroup. A coset of H is a left coset, s_0H, for some $s_0 \in G$. We remark that right cosets $Hs_0 = s_0(s_0^{-1}Hs_0)$ are left cosets of the (possibly different) subgroup $s_0^{-1}Hs_0$. If $C = s_0H$ is a coset, then $C^{-1}C = H$ and $CC^{-1}C = C$. With G_1 another group, a map $\alpha\colon H \to G_1$ is *affine* if $\alpha(rs^{-1}t) = \alpha(r)\alpha(s)^{-1}\alpha(t)$ for $r, s, t \in H$. This is equivalent to $H \to G_1$, $s \mapsto \alpha(s_0)^{-1}\alpha(s_0s)$ being a group homomorphism. Given a subset $A \subseteq G$, let $\mathrm{aff}(A)$ be the smallest coset containing A.

The *coset ring* of G, denoted $\Omega(G)$, is the smallest ring of subsets of G containing all cosets of all subgroups of G. Given $Y \subseteq G$, a map $\alpha\colon Y \to G_1$ is *piecewise affine* when Y is the finite disjoint union of sets $(Y_i)_{i=1}^n$ in $\Omega(G)$, and for each i, there is an affine map $\alpha_i\colon \mathrm{aff}(Y_i) \to G_1$ with $\alpha|_{Y_i} = \alpha_i|_{Y_i}$. See [3, Sections 2, 4] and [4, Section 1.2] for combinatorial details about cosets and $\Omega(G)$.

Now let G be a locally compact group, and let $\Omega_o(G)$ be the ring of sets generated by open cosets of G. As an open subgroup is also closed, the same applies to open cosets, and so every member of $\Omega_o(G)$ is clopen. The key lemma in [4] is Lemma 1.3 (ii), which states that, with G_1 another locally compact group, if $\alpha\colon Y \to G_1$ is piecewise affine, and α is continuous, and Y is open, then α has a continuous extension to $\overline{\alpha}\colon \overline{Y} \to G_1$. Furthermore, $\overline{Y}$ is open, and in the decomposition $\overline{Y} = \bigsqcup_i Y_i$, we may assume that each $Y_i \in \Omega_o(G)$, and for each i, there is an open coset C_i containing Y_i, and a continuous affine map $\alpha_i\colon C_i \to G_1$ which agrees with $\overline{\alpha}$ on Y_i. The use of this lemma is that it allows us to combine the algebraic property that α is piecewise affine with the topological property that α is continuous, and conclude that α is the "union" of continuous affine maps.

In the proof of [4, Lemma 1.3 (ii)], we have a coset K and subcosets $N_1, \ldots, N_k$ of infinite index, and it is claimed that if $Y = K \setminus \bigcup_j N_j$ with $\overline{Y}$ having non-empty interior, then $\overline{Y} = \overline{K} \setminus \bigcup_{j\in J} N_j$, where J is the collection of indices with

N_j having non-empty interior. This is not true. A counter-example from [10] exhibits a compact abelian group G_0 and an index 2 subgroup H_0 so that both H_0 and $H_1 = G_0 \setminus H_0$ have empty interior, and yet of course $G_0 = H_0 \sqcup H_1$. If we set $G = K = G_0 \times \mathbb{Z}$ and $N_i = H_i \times \{0\}$, then each N_i has infinite index in K, each has empty interior, $Y = K \setminus (N_0 \cup N_1) = G_0 \times (\mathbb{Z} \setminus \{0\})$ is already clopen, as is K, and yet $\overline{Y} \neq \overline{K}$. The same issue can be seen in the middle of [8, Section 4.5.2].

This counter-example does not mention (piecewise) affine maps, but we could simply let α be the identity. The moral seems to be that we chose a "silly" way to write α as a piecewise affine map. However, given an arbitrary piecewise affine map α which we happen to know is continuous, we need some argument to show that we can exhibit that α is piecewise affine in a "sensible" way. Cohen's original argument uses knowledge about the graph of α and then a delicate combinatorial argument to show that we can exhibit the graph using sets built from the graph itself (compare Theorem 3.2 below). This result can then be used to show that we can exhibit that α is piecewise affine using at least measurable sets (the subgroup H_0 in the example above is not measurable) which together with a measure-theoretic argument then yields an analogue of [4, Lemma 1.3 (ii)]. We will use exactly the same general approach, but adapted to possibly non-abelian groups.

Some of our argument closely follows Cohen's paper [1]. We must say that we find many of Cohen's arguments rather hard to follow. In particular, our key technical result, Proposition 3.9, is similar to the lemma on [1, pp. 223–224], the proof of which we do not understand. From our limited understanding, it seems clear, however, that this lemma of Cohen requires at least that every subgroup involved be normal (which is automatic if the groups are abelian!). Given that we need to check that all results hold for non-abelian groups, and that our central argument is entirely new, we have decided to give full details for all our results. We indicate in a number of places where we follow Cohen quite closely.

2 Initial setup of the problem

We fix locally compact groups G, H and a completely bounded homomorphism $\Phi\colon A(G) \to B(H)$. Following the proof of [4, Theorem 3.7], there is a continuous map $\alpha\colon H \to G_\infty$ with $\Phi(u)(s) = u(\alpha(s))$ for each $s \in H$. Here G_∞ is either the one-point compactification of G if G is not compact, or the disjoint union $G \sqcup \{\infty\}$ if G is compact. We extend each $u \in A(G)$ to a (continuous) function on G_∞ by setting $u(\infty) = 0$. Then $Y = \alpha^{-1}(G)$ is open in H. Under the additional hypothesis that G is amenable, $\alpha\colon Y \to G$ is piecewise affine if we regard H and G as just groups, with no topology. In what follows, we shall not use amenability again.

From now on, G and H will be arbitrary groups, with additional hypotheses stated as needed. Given $Y \subseteq H$ and $\alpha: Y \to G$, we shall in the sequel write $\alpha: Y \subseteq H \to G$. The *graph* of α is

$$\mathcal{G}(\alpha) = \{(s, \alpha(s)) : s \in Y\} \subseteq H \times G.$$

An extremely useful result is the following.

Lemma 2.1 ([4, Lemma 1.2]). *Let $\alpha: Y \subseteq H \to G$ be a map. Then*

$$\mathcal{G}(\alpha) \in \Omega(H \times G)$$

if and only if α is piecewise affine.

In the next section, we shall prove our main result, Theorem 3.2. In the following section, we apply this to show that α can be exhibited as a piecewise affine map with the component sets involved at least being Borel, Proposition 4.4. In the σ-finite case, a measure-theoretic argument then yields what we want and can be bootstrapped into a proof in the general case, Theorem 4.7.

3 Combinatorial lemma

We begin by making a non-standard, but useful, definition. (This definition is sometimes termed the "measure theory ring of sets", but we shall stick to our ad hoc definition for clarity.)

Definition 3.1. Let X be a set and $\mathcal{S}$ a collection of subsets of X. We say that $\mathcal{S}$ is a *relative ring of subsets* when $\mathcal{S}$ is closed under finite unions, intersections, and relative complements, in the sense that if $A, B \in \mathcal{S}$, then $A \setminus B \in \mathcal{S}$. This means that $\emptyset \in \mathcal{S}$, but perhaps X is not in $\mathcal{S}$.

Let G be a group and α a collection of subsets of G. Let $\mathcal{R}(\alpha)$ be the relative ring of subsets generated by α and all left translations of elements of α. Let $\mathcal{R}_2(\alpha)$ be the relative ring of subsets generated by α and all two-sided translates of α, that is, sets of the form sAt, where $A \in \alpha$, $s, t \in G$.

This section is devoted to proving the following result. As to why we work with two-sided translates and not just left translates, see Remark 3.16 below.

Theorem 3.2. *Let G be a group, and let $Y \in \Omega(G)$. Then there are subgroups $H_1, H_2, \dots, H_n$ in $\mathcal{R}_2(\{Y\})$ such that $Y \in \mathcal{R}(\{H_1, \dots, H_n\})$.*

We first of all collect some combinatorial lemmas. These results are similar in spirit to arguments found in [1, Section 3]. In the following, the empty intersection is by definition equal to X, and the empty union equal to $\emptyset$.

Lemma 3.3. *Let X be a set, let $\mathfrak{A}$ be a collection of subsets of X, and let B be in the relative ring generated by $\mathfrak{A}$. Then B is the finite (disjoint if we wish) union of sets of the form*

$$\bigcap_{i=1}^{n} A_i \cap \bigcap_{j=1}^{m} (X \setminus B_j) = \left(\bigcap_{i=1}^{n} A_i\right) \setminus \left(\bigcup_{j=1}^{m} B_j\right) \tag{3.1}$$

for some $n \geq 1$, $m \geq 0$, and some $A_i, B_j \in \mathfrak{A}$.

Proof. Let $\mathfrak{B}$ be the relative ring generated by $\mathfrak{A}$, so $B \in \mathfrak{B}$. Let $\mathfrak{B}'$ be the collection of finite unions of sets of the form (3.1) so that $\mathfrak{A} \subseteq \mathfrak{B}' \subseteq \mathfrak{B}$. We shall prove that $\mathfrak{B}'$ is a relative ring of sets so that $\mathfrak{B}' = \mathfrak{B}$, as claimed.

By definition, $\mathfrak{B}'$ is closed under unions. If $(P_k)_{k=1}^t$ and $(Q_l)_{l=1}^s$ are of the form (3.1), set $P = \bigcup_k P_k$ and $Q = \bigcup_l Q_l$; then

$$P \cap Q = (P_1 \cup \cdots \cup P_t) \cap (Q_1 \cup \cdots \cup Q_s) = \bigcup_{k,l} P_k \cap Q_l,$$

and clearly $P_k \cap Q_l$ is of the form (3.1). So $\mathfrak{B}'$ is closed under intersections. Furthermore,

$$P \setminus Q = P \cap (X \setminus Q) = P \cap (X \setminus Q_1) \cap \cdots \cap (X \setminus Q_s).$$

Thus, to show that $P \setminus Q \in \mathfrak{B}'$, it suffices to show that, say $P \cap (X \setminus Q_1) \in \mathfrak{B}'$. Let $Q_1 = \bigcap_{i=1}^n A_i \cap \bigcap_{j=1}^m (X \setminus B_j)$ so that

$$\begin{aligned} P \cap (X \setminus Q_1) &= P \cap \left(\bigcup_{i=1}^{n} (X \setminus A_i) \cup \bigcup_{j=1}^{m} B_j\right) \\ &= \bigcup_{i=1}^{n} (P \cap (X \setminus A_i)) \cup \bigcup_{j=1}^{m} (P \cap B_j) \\ &= \bigcup_{i=1}^{n} (P \setminus A_i) \cup \bigcup_{j=1}^{m} (P \cap B_j). \end{aligned}$$

As each $P \cap B_j \in \mathfrak{B}'$ and $\mathfrak{B}'$ is closed under unions, it remains to show that, say $P \setminus A_1 \in \mathfrak{B}'$, but $P \setminus A_1 = \bigcup_{k=1}^t P_k \setminus A_1$, so in fact, it remains to show that, say $P_1 \setminus A_1 \in \mathfrak{B}'$. If $P_1 = \bigcap_{i=1}^{n'} C_i \cap \bigcap_{j=1}^{m'} (X \setminus D_j)$, then

$$P_1 \setminus A_1 = \bigcap_{i=1}^{n'} C_i \cap \bigcap_{j=1}^{m'} (X \setminus D_j) \cap (X \setminus A_1)$$

and so indeed $P_1 \setminus A_1 \in \mathfrak{B}'$.

Having shown that $\mathfrak{B} = \mathfrak{B}'$, it is easy that each member of $\mathfrak{B}$ is a *disjoint* union of sets of the form (1) because, for any sets (A_i), we have that

$$\bigcup_{i=1}^{n} A_i = A_1 \cup (A_2 \setminus A_1) \cup (A_3 \setminus (A_1 \cup A_2)) \cup \cdots . \qquad \square$$

We shall use the following repeatedly, and so we pull it out as a lemma.

Lemma 3.4. *Let G be a group, and let $H_1, \ldots, H_n$ be subgroups. There are subgroups $K_1, \ldots, K_{n'}$, each a subgroup of some H_i, and with $[K_i : K_i \cap K_j] = 1$ or ∞ for all i, j, and with each H_i a finite union of cosets of the K_j. If the family of subgroups $\{H_i\}$ is closed under taking intersections, then the family $\{K_i\}$ is a subfamily of $\{H_i\}$.*

Proof. If, for some i, $[H_1 : H_1 \cap H_i]$ is not 1 or ∞, then $L = H_1 \cap H_i$ is a subgroup of H_1 of finite index, and so H_1 can be covered by finitely many translates of L. We can hence replace H_1 by L; notice that $[L : L \cap H_i] = 1$. We claim further that, by replacing H_1 by L, if previously $[H_1 : H_1 \cap H_j] \in \{1, \infty\}$, then we do not change this property.

Indeed, if $[H_1 : H_1 \cap H_j] = 1$, then $H_1 = H_1 \cap H_j$, and so also $L \subseteq H_j$, so $[L : L \cap H_j] = 1$. If $[H_1 : H_1 \cap H_j] = \infty$, then let $K = H_1 \cap H_j$ so that $L \cap H_j = H_1 \cap H_i \cap H_j = L \cap K$, and hence we have $L \cap K \leq L \leq H_1$. Towards a contradiction, suppose that $[L : L \cap H_j] < \infty$, so $[L : L \cap K] < \infty$. As L is of finite index in H_1, we can find $t_i \in H_1$ with $H_1 = \bigcup_{i=1}^{m} t_i L$, and we can find r_j in L with $L = \bigcup_{j=1}^{m'} r_j (L \cap K)$. Thus

$$H_1 = \bigcup_{i=1}^{m} t_i L = \bigcup_{i=1}^{m} \bigcup_{j=1}^{m'} t_i r_j (L \cap K),$$

and so certainly $H_1 = \bigcup_{i,j} t_i r_j K$, so $[H_1 : H_1 \cap H_j] \leq m m' < \infty$, a contradiction.

By performing this argument finitely many times, we may suppose that, for each i, $[H_1 : H_1 \cap H_i] \in \{1, \infty\}$. We now look at H_2 and apply the same argument, and so forth. This process could lead to repeats, and so possibly $n' \leq n$.

For the final remark, note that, by construction, each K_i is an intersection of some of the H_j, and thus $\{K_i\}$ is a subfamily of the $\{H_j\}$. $\square$

The following result is used extensively in [1, 4] and appears to have first been shown in [5]. Given the tools we now have, this is easy, so we give the proof. This proof is essentially Cohen's from [1].

Lemma 3.5. *Let G be a group. G cannot be written as the finite union of cosets of infinite index.*

Proof. Towards a contradiction, suppose that G is the finite union of cosets of subgroups $H_1, \dots, H_k$, where $[G : H_i] = \infty$ for each i. By Lemma 3.4, there are subgroups of these, say $K_1, \dots, K_{k'}$ with $[K_i : K_i \cap K_j] = 1$ or ∞ for all i, j, and still with each K_i of infinite index in G. As each H_j is the finite union of cosets of some K_i, it follows that G can be covered by finitely many cosets of the K_i, say $C_1, \dots, C_n$.

As K_1 is of infinite index in G, there is some coset of K_1 which is not a member of our covering, say $K = sK_1$. Then $K \cap K_i$ is a coset of $K_1 \cap K_i$ or is empty for each i. However, as G is covered by the C_i, also K is covered by $\{K \cap C_i\}$. If C_i is a coset of H_1, then $K \cap C_i = \emptyset$, so we conclude that K is covered by finitely many cosets of the subgroups $K_2, K_3, \dots, K_{k'}$. By translating by s^{-1}, also K_1 is covered by finitely many cosets of the subgroups $K_2, K_3, \dots, K_{k'}$.

We now complete the argument by using induction on the number of subgroups, k'. If we have only one subgroup, that is, $k' = 1$, the result is trivially true. The previous paragraph then provides the induction step. □

As the intersection of two cosets is again a coset, Lemma 3.3 immediately implies that every member of $\Omega(G)$ is the finite disjoint union of sets of the form $E_0 \setminus \bigcup_{i=1}^n E_i$, where each E_0 is a coset, and by replacing E_i with $E_0 \cap E_i$, we may suppose further that $E_i \subseteq E_0$ for each i. In fact, by using the arguments above, we can say more (again, see [1,3]).

Corollary 3.6. *Let G be a group. Every member of $\Omega(G)$ is either empty or is a finite disjoint union of sets of the form*

$$E_0 \setminus \bigcup_{i=1}^{n} E_i, \tag{3.2}$$

where each E_i is a coset, $E_i \subseteq E_0$, and E_i has infinite index in E_0 for each $i > 0$.

Proof. Let $Y \in \Omega(G)$, so in particular, Y is in the ring generated by some subgroups $H_1, \dots, H_n$ and their translates. By adding in intersections, we may suppose that the finite family $\{H_i\}$ is closed under intersections. By Lemma 3.4, we may suppose that $[H_i : H_i \cap H_j] = 1$ or ∞ for each i, j. By using this lemma, the resulting family $\{H_i\}$ may no longer be closed under intersections, but if H is an intersection of some of the H_i, then H is at least a finite union of cosets of some H_j.

By Lemma 3.3, Y is the disjoint union of say n sets of the form $E_0 \setminus \bigcup_{i=1}^m E_i$, where E_0 is a coset of some finite intersection of the H_j, and each E_i is a coset of some H_j. Hence E_0 is a finite union of cosets of the H_j, and so (by maybe increasing n) we may suppose that E_0 actually is a coset of some H_j. As before, by replacing E_i by $E_0 \cap E_i$, we may suppose that E_i is a subcoset of E_0. As $[H_i : H_i \cap H_j] = \infty$ unless $H_i \subseteq H_j$, if E_i has finite index in E_0, then E_i and E_0 must be cosets of the same subgroup, and as $E_i \subseteq E_0$, we must have $E_0 = E_i$, a case which may be ignored. □

We now depart from the presentation of [1] and collect some further lemmas which will be used later.

Lemma 3.7. *Let G be a group, and let $H_1, \ldots, H_n$ be subgroups of infinite index in G. Let $(K_i)_{i=1}^N$ be cosets of the H_j, and let $C = \bigcup_i K_i$. There is m and $t_1, \ldots, t_m \in G$ with $\bigcap_i t_i C = \emptyset$.*

Proof. Notice that

$$\bigcap_{i=1}^m t_i C = \bigcap_{i=1}^m \bigcup_j t_i K_j = \bigcup \Big\{ \bigcap_{i=1}^m t_i K_{j(i)} \Big\},$$

where the union is over all functions $j : \{1, \ldots, m\} \to \{1, \ldots, N\}$. Thus we need to find t_i so that $\bigcap_{i=1}^m t_i K_{j(i)} = \emptyset$ for any such function j. In what follows, suppose that K_j is the coset $s_j H_{k(j)}$ for each j.

Set $t_1 = e$, the identity. We claim that there is t_2 with $K_j \cap t_2 K_j = \emptyset$ for all j. Indeed, if not, then for each t_2, there is j with $s_j H_{k(j)} = t_2 s_j H_{k(j)}$ (as cosets are either equal or disjoint). Equivalently, for each t_2, there is j with $s_j^{-1} t_2 s_j \in H_{k(j)}$, so $t_2 \in s_j H_{k(j)} s_j^{-1}$. As each subgroup $s_j H_{k(j)} s_j^{-1}$ is of infinite index in G, this shows that G is covered by a finite union of subgroups of infinite index, a contradiction.

Suppose we have chosen $t_1, \ldots, t_p$ with the property that $t_1 K_{j(1)} \cap \cdots \cap t_p K_{j(p)}$ is only (possibly) non-empty when the $j(i)$ are all distinct. We have already shown this is true for $p = 2$. To show that the claim holds for $p + 1$, it suffices to find t_{p+1} with $t_i K_j \cap t_{p+1} K_j = \emptyset$ for all $i \leq p$ and all j. This is sufficient, for if we have $j(1), \ldots, j(p+1)$ not all distinct, say $j(k) = j(k')$ for $k < k'$, then if $k' \leq p$, we know already that $\bigcap_{i=1}^p t_i K_{j(i)} = \emptyset$, while if $k' = p + 1$, then by construction, $t_k K_{j(k)} \cap t_{p+1} K_{j(k)} = \emptyset$, so certainly $\bigcap_{i=1}^{p+1} t_i K_{j(i)} = \emptyset$.

To find t_{p+1}, we again proceed by contradiction and suppose that, for each $t \in G$, there are some i, j with $t_i K_j = t K_j$ so that $t_i s_j H_{k(j)} = t s_j H_{k(j)}$, so $s_j^{-1} t_i^{-1} t s_j \in H_{k(j)}$. That is, $t \in \bigcup_{i,j} t_i s_j H_{k(j)} s_j^{-1}$ which is again a finite union of cosets of infinite index, which cannot cover G.

So, by induction, our claim holds for all p. Thus $\bigcap_{i=1}^{p} t_i K_{j(i)}$ can only possibly be non-empty when the $j(i)$ are all distinct, but there are only N many choices, and so if $p > N$, we have shown our claim. □

Lemma 3.8. *Let G be a group, H a subgroup, and $H_1, \ldots, H_n$ subgroups with $[H : H \cap H_i] = \infty$ for each i. Let C be a union of finitely many cosets of the H_i. There are m and $t_1, \ldots, t_m \in H$ with $\bigcap_i C t_i = \emptyset$.*

Proof. The proof is similar to the previous one; but note here we need to choose t_i in H not G. Let our cosets be C_j for $j = 1, \ldots, N$. Again, we need to find t_i with $C_{j(1)} t_1 \cap \cdots \cap C_{j(m)} t_m = \emptyset$ for any choices $j(i)$. Set $t_1 = e$.

Suppose we have $t_1, \ldots, t_p$ so that if $j(1), \ldots, j(p)$ are not distinct, then

$$\bigcap_{i=1}^{p} C_{j(i)} t_i = \emptyset.$$

Proceeding by induction, $t_{p+1} \in H$ needs to satisfy that $C_j t_i \cap C_j t_{p+1} = \emptyset$ for all i, j. If no such t_{p+1} exists, then for all $t \in H$, there are some i, j with

$$C_j t_i \cap C_j t \neq \emptyset.$$

If $C_j = sH_k$, say, then $sH_k t_i \cap sH_k t \neq \emptyset$, so there are $a, b \in H_k$ with $sat_i = sbt$, so $at_i = bt$, so $tt_i^{-1} = b^{-1}a \in H_k$, so $t \in H_k t_i$. Thus $H \subseteq \bigcup_{i,k} H_k t_i$, but as each $t_i \in H$, we have $H \cap H_k t_i = (H \cap H_k) t_i$, and so $H = \bigcup_{i,k} (H \cap H_k) t_i$. As each $H \cap H_k$ is of infinite index in H, this is a contradiction. □

We now start on our proof of Theorem 3.2. We start with $Y \in \Omega(G)$, so there are subgroups $H_1, \ldots, H_n$ so that Y is in the relative ring of sets generated by the H_i (if necessary, we can choose one of the H_i to be G). We may suppose that the family $\{H_i\}$ is closed under intersection. Then apply Lemma 3.4 to suppose that $[H_i : H_i \cap H_j] = 1$ or ∞ for all i, j, and that if H is the intersection of some of the H_i, then H is a finite disjoint union of cosets of some H_j. Using Lemma 3.3 as in the proof of Corollary 3.6, Y is a finite union of sets $L_1, \ldots, L_m$, where

$$L_i = E_0^{(i)} \setminus \bigcup_{j=1}^{n_i} E_j^{(i)}, \tag{3.3}$$

where each $E_j^{(i)}$ is a coset of some H_k, and $E_j^{(i)}$ is a subcoset of infinite index in $E_0^{(i)}$ for each i and $j > 0$.

Our proof will be an induction, with the base case provided by the following proposition.

Proposition 3.9. *With notation as above, if H_i is not contained in any other H_j, then there is a subgroup H which contains H_i as a finite-index subgroup (possibly $H = H_i$) such that $H \in \mathcal{R}_2(\{Y\})$ and such that $Y \in \mathcal{R}(\{H\} \cup \{H_j : j \neq i\})$.*

Given this, we can now prove Theorem 3.2.

Proof of Theorem 3.2. We can form a directed acyclic graph (DAG, see [6] for example) with vertices the subgroups H_i, and with a directed edge from H_i to H_j when $H_i \supseteq H_j$ and there is no other H_k with $H_i \supseteq H_k \supseteq H_j$. We shall say that H_i is *top level* if H_i is not contained in any other H_j, that is, there is no edge into H_i. The *depth* of a DAG is the length of the longest directed path in the DAG.

For each top level H_i, let H be given by Proposition 3.9. For any $i \neq j$ as $[H_i : H_i \cap H_j] = 1$ or ∞, also $[H : H \cap H_j] = 1$ or ∞; see Lemma 3.15 below. Also, if $H_i \cap H_j$ is the finite union of cosets of H_k, then so is $H \cap H_j$; see Lemma 3.15 below. Hence we may replace H_i by H and not change any of our assumptions. Do this for all top level H_i so that $Y \in \mathcal{R}(\{H_i\})$ and each top-level H_i is in $\mathcal{R}_2(\{Y\})$.

We shall give a proof by induction on the depth of the DAG. Notice that Proposition 3.9, and the previous paragraph, shows that the result is true when the DAG has depth 0 (that is, all subgroups are top level).

Let H be some top level H_i. Let $Y = \bigsqcup_i L_i$ as before (see (3.3)), and reorder these so that $E_0^{(i)}$ is a coset of H for $i \leq n_0$, and not for $i > n_0$. Define

$$Y_0 = \bigcup_{i=1}^{n_0} E_0^{(i)} \setminus Y \subseteq \bigcup_{i=1}^{n_0} E_0^{(i)}.$$

From the form that each L_i is written in, namely that $E_j^{(i)}$ is a coset of infinite index in $E_0^{(i)}$, it is clear that $E_j^{(i)}$ is not a coset of H for $j \geq 1$. From this, it follows that $Y_0 \in \mathcal{R}(\{H \cap H_i\} \setminus \{H\})$. For each i, either $H \cap H_i = H$ which we remove, or $H \cap H_i$ is a union of cosets of some H_k, where necessarily $H_k \subseteq H$. Thus actually $Y_0 \in \mathcal{R}(\{H_k : H_k \subsetneq H\})$.

Now the family $\{H_k : H_k \subsetneq H\}$ forms a subgraph of our DAG; in fact, it is DAG "underneath" H (all the vertices which have a path leading to them from H). Thus it is of smaller depth, so by induction, there are subgroups H'_j in $\mathcal{R}_2(\{Y_0\})$ with $Y_0 \in \mathcal{R}(\{H'_j\})$. Notice that, as $Y, H \in \mathcal{R}_2(\{Y\})$, also $Y_0 \in \mathcal{R}_2(\{Y\})$, and so also each $H'_j \in \mathcal{R}_2(\{Y\})$.

Next, reorder so that H_i is top level for $i \leq n_0$, and not for $i > n_0$, so if $i > n_0$, the subgroup H_i is contained in some H_j with $j \leq n_0$. For $i \leq n_0$, let A_i be the union of the sets $E_0^{(k)}$ which are cosets of H_i. Set $B_i = A_i \setminus Y$, and set $C = Y \setminus \bigcup_{i \leq n_0} A_i$. We have shown that, for each i, there are subgroups H'_j in $\mathcal{R}(\{Y\})$ with $B_i \in \mathcal{R}(\{H'_j\})$.

We now consider C. As each $E_0^{(i)}$ is a coset of some H_j, either $E_0^{(i)} \subseteq A_j$ for some j, or otherwise $E_0^{(i)}$ is a coset of some H_j which is not top level. Since each $E_j^{(i)}$ is a subcoset of $E_0^{(i)}$, we see that C is contained in

$$\bigcup\{E_0^{(i)} : E_0^{(i)} \text{ coset of some } H_j \text{ which is not top level}\},$$

and so $C \in \mathcal{R}(\{H_i : H_i \text{ not top level}\})$. The DAG given by removing all top level subgroups has smaller depth, and so again by induction, there are subgroups H_j'' in $\mathcal{R}_2(\{C\})$ with $C \in \mathcal{R}(\{H_j''\})$. Notice that

$$C \in \mathcal{R}_2(\{Y, A_i\}) \subseteq \mathcal{R}_2(\{Y, H_i : i \leq n_0\}) = \mathcal{R}_2(\{Y\}),$$

and so each $H_j'' \in \mathcal{R}_2(\{Y\})$.

Let α be the collection of all the H_j', the H_j'', and the top level H_i, so each subgroup in α is in $\mathcal{R}_2(\{Y\})$. Then $A_i, B_i, C \in \mathcal{R}(\alpha)$. As $A_i \setminus B_i = A_i \cap Y$, we see that

$$C \cup \bigcup_{i \leq n_0} (A_i \setminus B_i) = \Big(Y \setminus \bigcup_{i \leq n_0} A_i\Big) \cup \bigcup_{i \leq n_0} (A_i \cap Y) = Y.$$

Thus also $Y \in \mathcal{R}(\alpha)$, which completes the proof. □

Thus it remains to show Proposition 3.9.

Definition 3.10. Given the subgroups (H_i), and $A \subseteq G$ some subset, we shall say that a coset sH_j is *big in* A if $A \cap sH_j$ cannot be covered by finitely many cosets of subgroups in $\{H_i : i \neq j\}$.

Let us make some easy remarks about this definition, which we put into a lemma for future reference.

Lemma 3.11. *Let (H_i) be subgroups as above.*

(1) *If $A \subseteq B$ and sH_j is big in A, then it is big in B.*

(2) *With $A = \bigcup_{i=1}^n A_i$, we have that sH_j is big in A if and only if it is big in some A_i.*

Proof. (1) is clear. For (2), if sH_j is not big in any A_i, so $sH_j \cap A_i$ can be covered, and hence so can A (as A is a finite union); the converse follows as $A_i \subseteq A$ for each i. □

In the following results, we state the result for the subgroup H_1, but this is merely for notational convenience, as clearly there is nothing special about H_1 as compared to any other H_i.

Lemma 3.12. *With $Y = \bigsqcup_i L_i$ as before, we have that sH_1 is big in Y if and only if some L_i is of the form $E_0 \setminus \bigcup_j E_j$ with $E_0 = sH_1$. Furthermore, in this case, the choice of i is unique.*

Proof. If $L_i = sH_1 \setminus \bigcup E_j$ and yet sH_1 is not big in L_i, then $sH_1 \cap L_i = L_i$ can be covered by finitely many cosets of infinite index in H_1. Union these cosets with the $\{E_j : j > 0\}$ and we have covered all of sH_1 by finitely many cosets of infinite index in H_1, a contradiction. So sH_1 is big in L_i and hence big in Y, by Lemma 3.11 (1).

If sH_1 is big in Y, then by Lemma 3.11 (2), we have that sH_1 is big in some L_i. If $L_i = E_0 \setminus \bigcup_j E_j$, then towards a contradiction, suppose that E_0 is not sH_1. If E_0 is some other coset of H_1, then $sH_1 \cap E_0 = \emptyset$, so certainly sH_1 is not big in L_i. So E_0 is a coset of some other subgroup H_j, and so $sH_1 \cap H_j$ is either empty (again, not possible) or is a coset of $H_1 \cap H_j$ which has infinite index in H_1. Then $sH_1 \cap L_i$ is contained in a coset of infinite index in sH_1 and so is covered, so sH_1 is not big in L_i, a contradiction.

To show uniqueness, let each L_i have the form (3.3). If $E_0^{(i)} = E_0^{(j)} = sH_1$, then if $i \neq j$, then as L_i and L_j are disjoint, we must have that

$$\bigcup_{k\geq 1} E_k^{(i)} \cup \bigcup_{k\geq 1} E_k^{(j)}$$

is all of sH_1, a contradiction as these are cosets of infinite index. □

Let C be the union of all $E_0^{(i)}$ which are cosets of H_1, so by the lemma, C is the union of all cosets of H_1 which are big in Y. Define

$$\mathcal{B} = \left\{ \bigcup_{i=1}^{n} s_i H_1 : \text{there exists } A \in \mathcal{R}(\{Y\}) \text{ so that } sH_1 \text{ is big in } A \text{ if and only if } sH_1 = s_i H_1 \text{ for some } i \right\}.$$

That is, $\mathcal{B}$ is the collection of sets B which are finite unions of cosets of H_1, with the given property: there is $A \in \mathcal{R}(\{Y\})$ such that a coset sH_1 is big in A if and only if $sH_1 \subseteq B$.

Lemma 3.13. $\mathcal{B} = \mathcal{R}(\{C\})$.

Proof. To get a handle on $\mathcal{B}$, we need some information about $\mathcal{R}(\{Y\})$. By Lemma 3.3, every $A \in \mathcal{R}(\{Y\})$ is of the form

$$A = \left(\bigcap_{i=1}^{n} s_i Y\right) \setminus \left(\bigcup_{j=1}^{m} t_j Y\right),$$

where $n \geq 1$ and $s_i, t_j \in G$. (This follows as if $\mathfrak{A}$ is left-invariant, so will be $\mathcal{R}(\mathfrak{A})$.) We wish to know when sH_1 is big in A, in terms of the form $Y = \bigsqcup L_i$ as in equation (3.3).

Let us think about these two parts individually. Consider

$$Y \cap tY = \bigsqcup_{i,j} L_i \cap tL_j = \bigsqcup_{i,j} \Big((E_0^{(i)} \cap tE_0^{(j)}) \setminus \Big(\bigcup_k E_k^{(i)} \cup \bigcup_l tE_l^{(j)} \Big) \Big).$$

As argued in the "uniqueness" part of Lemma 3.12, either $E_0^{(i)} = tE_0^{(j)} = sH_1$, or $E_0^{(i)} \cap tE_0^{(j)} \cap sH_1$ is either empty or is a coset of infinite index in sH_1. It then follows from Lemma 3.12 that sH_1 is big in $Y \cap tY$ if and only if there are (unique) i, j with $E_0^{(i)} = tE_0^{(j)} = sH_1$. A similar argument now shows that sH_1 is big in $\bigcap s_i Y$ if and only if, for each i, there is a (necessarily unique) j with $s_i E_0^{(j)} = sH_1$.

By Lemma 3.11 (2), we see that sH_1 is big in $\bigcup t_j Y$ if and only if there is some j with sH_1 big in $t_j Y$, if and only if, by Lemma 3.12, there is j and a (necessarily unique) k with $sH_1 = t_j E_0^{(k)}$.

Let $B_0 = \bigcap s_i Y$ and $B_1 = \bigcup t_j Y$, so $A = B_0 \setminus B_1$.

- If sH_1 is not big in B_0, then it is not big in A by Lemma 3.11 (1).
- If sH_1 is big in B_0 and not big in B_1, then as $B_0 = A \cup B_1$, also sH_1 is big in A by Lemma 3.11 (2).
- If sH_1 is big in both B_0 and B_1, then from above, we know that $sH_1 \cap B_0$ is equal to sH_1 with a finite union of cosets of infinite index removed, while $sH_1 \cap B_1$ contains a set of the form sH_1 with a finite union of cosets of infinite index removed. Thus $sH_1 \cap (B_0 \setminus B_1)$ is contained in a finite union of cosets of infinite index, so sH_1 is not big in $A = B_0 \setminus B_1$.

In conclusion, sH_1 is big in A if and only if sH_1 is big in B_0 but not big in B_1. Now sH_1 is big in B_0 if and only if $sH_1 \subseteq s_i C$ for all i, that is, $sH_1 \subseteq \bigcap s_i C$. Also, sH_1 is big in B_1 exactly when $sH_1 \subseteq t_j C$ for some j, that is, $sH_1 \subseteq \bigcup t_j C$. Thus

$$\bigcup \{ sH_1 : sH_1 \text{ big in A} \} = \bigcap s_i C \setminus \bigcup t_j C, \tag{3.4}$$

as all these sets are unions of cosets of H_1. This shows that $\mathcal{B} \subseteq \mathcal{R}(\{C\})$.

To show the converse, we simply observe that, by Lemma 3.3, any member of $\mathcal{R}(\{C\})$ is of the form given by the right-hand side of equation (3.4) for some (s_i) and (t_j), and so the associated A will be a member of $\mathcal{R}(\{Y\})$, showing that $\mathcal{R}(\{C\}) \subseteq \mathcal{B}$. □

Proposition 3.14. *There is a subgroup H, which is in $\mathcal{B}$ and so a finite union of cosets of H_1, such that any element of $\mathcal{B}$ is a finite union of cosets of H.*

Proof. Every member of $\mathcal{B}$ is a finite union of cosets of H_1. Let

$$H = s_1 H_1 \sqcup \cdots \sqcup s_n H_1 \in \mathcal{B}$$

be chosen with $n > 0$ minimal (so H is the disjoint union of n cosets of H_1, and any member of $\mathcal{B}$ is the disjoint union of at least n cosets of H_1). By translating, we may suppose that $s_1 = e$, the identity. We will show that H is a subgroup.

If $A = \bigsqcup t_i H_1 \in \mathcal{B}$, then $A \cap H \in \mathcal{B}$ and so is either empty or the union of at least n cosets of H_1. As $A \cap H \subseteq H$, we must have $H = A \cap H$ or $A \cap H = \emptyset$. In particular, for each s, either $sH \cap H = H$ or $sH \cap H = \emptyset$. If $s \in H$, then as $e \in H$, also $s \in sH$, so $s \in H \cap sH$, so $H \cap sH = H$. Thus $HH \subseteq H$. Also, $e \in H \cap s^{-1}H$, and so $H \cap s^{-1}H = H$, so in particular,

$$s^{-1} = s^{-1}e \in s^{-1}H \subseteq H,$$

and we conclude that H is a subgroup.

Given $A \in \mathcal{B}$ and $s \in G$, notice that $sH \cap A = s(H \cap s^{-1}A)$ is either empty or equal to sH because $s^{-1}A \in \mathcal{B}$. It follows that A is a (necessarily finite) union of cosets of H. □

Lemma 3.15. *Let H be a subgroup containing H_1 with $[H : H_1] < \infty$. For each $i > 1$, we have that $[H : H \cap H_i] = \infty$ or 1, and that $H \cap H_i$ is a finite union of cosets of some H_k.*

Proof. For $i > 1$, we know that either

$$[H_1 : H_1 \cap H_i] = 1 \text{ or } \infty.$$

If $[H_1 : H_1 \cap H_i] = 1$, then $H_i \subseteq H_1 \subseteq H$. Otherwise, $[H_1 : H_1 \cap H_i] = \infty$, and we claim that also $[H : H \cap H_i] = \infty$. If not, then $H = \bigcup_{i=1}^n s_i(H \cap H_i)$ say, and as $H_1 \leq H$, it follows that $H_1 = H_1 \cap H = \bigcup H_1 \cap s_i(H \cap H_i)$. For each i, either $H_1 \cap s_i(H \cap H_i)$ is empty or is a coset of

$$H_1 \cap (H \cap H_i) = H_1 \cap H_i,$$

and so $[H_1 : H_1 \cap H_i] \leq n$, a contradiction.

Let $H_1 \cap H_i$ be a finite union of cosets of some H_k, this being one of our properties of the family $\{H_i\}$. As H_1 is finite index in H, we have $H = \bigcup_{j=1}^n s_j H_1$ say. Then $H \cap H_i = \bigcup_j s_j H_1 \cap H_i$, and for each j, either $s_j H_1 \cap H_i$ is empty or is a coset of $H_1 \cap H_i$, which is a finite union of cosets of H_k. Thus $H \cap H_i$ is also a finite union of cosets of H_k. □

We can now complete the proof of Proposition 3.9.

Proof of Proposition 3.9. Our hypothesis is that H_i is not contained in any other H_j. We have been supposing, by reordering, that $i = 1$. Let H be given by Proposition 3.14. We need to show that $H \in \mathcal{R}_2(\{Y\})$ and $Y \in \mathcal{R}(\{H\} \cup \{H_j : j \neq 1\})$.

As $H \in \mathcal{B}$, there is some $A_0 \in \mathcal{R}(\{Y\}) \subseteq \mathcal{R}(\{H_j\})$ so that sH_1 is big in A_0 if and only if $sH_1 \subseteq H$. Then A_0 has the form

$$A_0 = \bigsqcup_i \Big(F_0^{(i)} \setminus \bigcup_j F_j^{(i)} \Big), \tag{3.5}$$

where each $F_j^{(i)}$ is some coset of some H_k, and each $F_j^{(i)}$, $j > 0$, is a subcoset of infinite index in $F_0^{(i)}$. Then sH_1 is big in A_0 if and only if $sH_1 = F_0^{(i)}$ for some i. It follows that $H \setminus A_0$ is contained in the union of (1) $F_j^{(i)}$, where $F_0^{(i)}$ is a coset of H_1, and (2) $F_0^{(i)}$ a coset of some H_j for $j > 1$. So $H \setminus A_0$ is contained in a finite union of cosets of $\{H \cap H_j : j > 1\}$. By Lemma 3.15, $H \cap H_j$ is of infinite index in H for $j > 1$, and so we can apply Lemma 3.7 to H to find (t_i) in H with

$$\emptyset = \bigcap_i t_i (H \setminus A_0) = H \setminus \bigcup_i t_i A_0.$$

So if we set $B_0 = \bigcup_i t_i A_0$, then $H \subseteq B_0$ and $B_0 \in \mathcal{R}(\{Y\})$.

We claim that sH_1 is big in B_0 if and only if $sH_1 \subseteq H$, which is equivalent to $s \in H$. As $H \subseteq B_0$, "if" is clear. To show "only if", suppose sH_1 is big in B_0, so by Lemma 3.11 (2), sH_1 is big in $t_i A_0$ for some i, so $t_i^{-1} sH_1$ is big in A_0, so $t_i^{-1} sH_1 \subseteq H$, so $sH_1 \subseteq t_i H = H$.

B_0 has the same form as in equation (3.5), so again $sH_1 \subseteq H$ if and only if some $F_0^{(i)}$ is equal to sH_1. Thus, again, we see that B_0 is contained in the union of H and cosets of $\{H_j : j > 1\}$, say $B_0 \subseteq H \cup \bigcup_{i=1}^N A_i$, so each A_i is a coset of some H_j, $j > 1$. With $C = \bigcup_i A_i$, by Lemma 3.8, there are $t_1, \dots, t_m \in H$ with $\bigcap_i C t_i = \emptyset$. As $B_0 \subseteq H \cup C$,

$$\bigcap_i B_0 t_i \subseteq \bigcap_i (H \cup C) t_i = \bigcap_i (H \cup C t_i) = H \cup \bigcap_i C t_i = H.$$

Thus clearly $H = \bigcap_i B_0 t_i$, and so $H \in \mathcal{R}_2(\{Y\})$.

To finish the proof of Proposition 3.9, it remains to show that

$$Y \in \mathcal{R}(\{H\} \cup \{H_j : j \neq 1\}).$$

By Lemma 3.13, we have that $\bigcup \{E_0^{(i)} : E_0^{(i)} \text{ a coset of } H_1\}$ is a union of cosets of H. Reorder so that $E_0^{(1)} \cup \cdots \cup E_0^{(k)} = sH$, say, so that

$$\bigcup_{i=1}^k E_0^{(i)} \setminus \bigcup_j E_j^{(i)} = sH \setminus \bigcup_{i=1}^k \bigcup_j E_j^{(i)}.$$

Notice that each $E_j^{(i)}$ is a coset of some H_t for some $t > 1$. As H_1 is not contained in any other H_k, no $E_j^{(i)}$, with $j > 0$, is a coset of H_1. Thus, in this way, we can replace every usage of a coset of H_1 by a coset of H, so proving the claim and completing the proof. □

Remark 3.16. It was only in the final step of the previous proof that we started to work with right translations, as well as left translations. This seems necessary, as the following example shows. Consider $\mathbb{F}_2$ the free group with generators a, b, and set $H_1 = \langle a \rangle = \{a^n : n \in \mathbb{Z}\}$ and $H_2 = bH_1b^{-1}$. Then H_1, H_2 are subgroups and $H_1 \cap H_2 = \{e\}$, so the family $\{H_1, H_2\}$ satisfies our assumptions.

Now let $Y = H_1 \sqcup b^{-1}H_2$ which is the disjoint union of cosets of the H_i. Then $Y = \langle a \rangle \sqcup \langle a \rangle b^{-1}$. For $x \in \mathbb{F}_2$, let $x = ya^n$, where y is a reduced word in a, b which does not end in a, a^{-1}, and $n \in \mathbb{Z}$, so that $x\langle a\rangle = y\langle a\rangle$. Thus $xY = y\langle a\rangle \sqcup y\langle a\rangle b^{-1}$, and so xY is either equal to Y or disjoint from Y.

We conclude that $\mathcal{R}(\{Y\})$ just consists of finite disjoint unions of left translates of Y. In particular, neither H_1 nor H_2 is in $\mathcal{R}(\{Y\})$.

4 Application to completely bounded maps

We now use Theorem 3.2 to give a new proof of the main result of [4]. We are now following the end of [1, Section 4] fairly closely, but again with more details provided and changes made from the abelian setting.

Lemma 4.1. *Let G, H be topological spaces, $Y \subseteq H$ a subset, and $\alpha: Y \to G$ a map which is continuous when Y has the subspace topology. Suppose there is a continuous map $\overline{\alpha}: \overline{Y} \to G$ extending α. Let $\Gamma = \mathcal{G}(\alpha) \subseteq H \times G$, the graph of α. If $\Gamma = U \cap C$ for some open U and closed C in $H \times G$, then $Y = \overline{Y} \cap V$ for some open $V \subseteq H$. In particular, Y is Borel.*

Proof. As $\Gamma \subseteq C$, also $\overline{\Gamma} \subseteq C$, and as $\Gamma \subseteq U$, also $\Gamma \subseteq \overline{\Gamma} \cap U \subseteq C \cap U = \Gamma$, so we conclude that $\Gamma = \overline{\Gamma} \cap U$.

We claim that $\mathcal{G}(\overline{\alpha}) = \overline{\Gamma}$, which follows by continuity of $\overline{\alpha}$. Indeed, given $y \in \overline{Y}$, let (y_i) be a net in Y converging to y so that $\alpha(y_i) = \overline{\alpha}(y_i) \to \overline{\alpha}(y)$ by continuity, and hence $(y_i, \alpha(y_i)) \to (y, \overline{\alpha}(y))$. Hence $\mathcal{G}(\overline{\alpha}) \subseteq \overline{\Gamma}$. Given $(y, x) \in \overline{\Gamma}$, let $(y_i, \alpha(y_i))$ be a net in Γ converging to (y, x), so $y_i \to y, \alpha(y_i) \to x$, so continuity of $\overline{\alpha}$ ensures that $\overline{\alpha}(y) = x$. Thus $\overline{\Gamma} \subseteq \mathcal{G}(\overline{\alpha})$.

Let $V_0 = \{y \in \overline{Y} : (y, \overline{\alpha}(y)) \in U\}$. Given $y \in V_0$, as $U \subseteq H \times G$ is open, by the definition of the product topology, there are $W_1 \subseteq H$ and $W_2 \subseteq G$ both open, with $y \in W_1, \overline{\alpha}(y) \in W_2$ and $W_1 \times W_2 \subseteq U$. Then $\overline{\alpha}^{-1}(W_2)$ is open in $\overline{Y}$, so there is $W_0 \subseteq H$ open with $W_0 \cap \overline{Y} = \overline{\alpha}^{-1}(W_2)$. Given $x \in W_0 \cap W_1 \cap \overline{Y}$, we have that $x \in \overline{\alpha}^{-1}(W_2)$, so $\overline{\alpha}(x) \in W_2$, and $x \in W_1$, so $(x, \overline{\alpha}(x)) \in U$, so $x \in V_0$. Also, $y \in W_0 \cap W_1 \cap \overline{Y}$. Thus V_0 is open in $\overline{Y}$, as we have shown that each point y has an open neighbourhood, namely $W_0 \cap W_1 \cap \overline{Y}$.

Let $V \subseteq H$ be open with $V \cap \overline{Y} = V_0$. Given $y \in V_0$, so $(y, \overline{\alpha}(y)) \in U$, also $(y, \overline{\alpha}(y)) \in \mathcal{G}(\overline{\alpha}) = \overline{\Gamma}$, and so $(y, \overline{\alpha}(y)) \in U \cap \overline{\Gamma} = \Gamma$, and so $y \in Y$. Conversely, if $y \in Y$, then $(y, \overline{\alpha}(y)) = (y, \alpha(y)) \in \Gamma = \overline{\Gamma} \cap U$, and so $y \in V_0$. Thus $\overline{Y} \cap V = Y$ as required. □

Proposition 4.3 below is implicitly assumed in the proof of [4, Lemma 1.3 (ii)], but we do not see why it follows immediately "by uniformity of the topology"; compare the argument on [1, p. 223], which we follow.

Lemma 4.2. *Let $L = E_0 \setminus \bigcup_{k=1}^n E_k$ be of the form (3.2), so E_0 is a coset and each E_k is a subcoset of infinite index. For any N, there are $a_1, \ldots, a_N \in L$ with, for $i \neq j$, $a_i^{-1} a_j \notin E_k^{-1} E_k$ for any $k \geq 1$.*

Proof. We first show that this is true for $N = 2$. If not, then for all $a, b \in L$, there is some k with $a^{-1} b \in E_k^{-1} E_k$. That is, for each $a \in L$, we have

$$L \subseteq \bigcup_{k=1}^{n} a\ E_k^{-1} E_k.$$

As $E_k^{-1} E_k$ is the subgroup which E_k is a coset of, this shows that L is contained in the finite union of cosets of infinite index, and so E_0 is contained in some finite union of cosets of infinite index, a contradiction.

We now proceed by induction. Suppose the claim holds for $N \geq 2$, but does not hold for $N + 1$. Then, given any $a_1, \ldots, a_N$ satisfying the claim, we cannot find a_{N+1} satisfying the claim so that, for any $b \in L$, we have that $a_i^{-1} b \in E_k^{-1} E_k$ for some i, k. That is,

$$L \subseteq \bigcup_{i=1}^{N} \bigcup_{k=1}^{n} a_i E_k^{-1} E_k.$$

Again, it follows from this that E_0 is contained in some finite union of cosets of infinite index, which is a contradiction. □

Proposition 4.3. *Let G, H be locally compact groups, and let*

$$L = E_0 \setminus \bigcup_{k=1}^{n} E_k \subseteq H$$

be of the form (3.2). Let $\alpha: L \to G$ be a map which is continuous on L for the subspace topology and is the restriction of some affine map $\psi: E_0 \to G$. Then ψ is continuous.

Proof. Choose $a_1, \ldots, a_{n+1} \in L$ using the lemma. We claim that,

$$\text{for any } x, y \in E_0, \text{ there is } k \text{ with } xy^{-1}a_k \in L.$$

Indeed, if not, then as $xy^{-1}a_k \in E_0$, we must have that $xy^{-1}a_k \in \bigcup_{i=1}^n E_i$ for each k. By the pigeonhole principle, there is i and $1 \leq r < s \leq n+1$ with

$$xy^{-1}a_r, xy^{-1}a_s \in E_i.$$

Thus $(xy^{-1}a_r)^{-1}xy^{-1}a_s = a_r^{-1}a_s \in E_i^{-1}E_i$, a contradiction.

Let U be an open neighbourhood of e in G, and choose an open symmetric neighbourhood V of e with $VV \subseteq U$. Then $V\alpha(a_k)$ is an open neighbourhood of $\alpha(a_k)$, and so, by continuity of α, and the definition of the subspace topology, there is an open V_k in H with

$$V_k \cap L = \alpha^{-1}(V\alpha(a_k)).$$

Then $a_k \in V_k \cap L$, and given $x, y \in V_k \cap L$, we see that $\alpha(x) = v_0\alpha(a_k)$ and $\alpha(y) = v_1\alpha(a_k)$ for some $v_0, v_1 \in V$. Then

$$\alpha(x)\alpha(y)^{-1} = v_0\alpha(a_k)\alpha(a_k)^{-1}v_1^{-1} = v_0v_1^{-1} \in VV \subseteq U.$$

Now set $V_0 = \bigcap_{k=1}^m V_k a_k^{-1}$ an open neighbourhood of e in H. Let $x, y \in E_0$ with $xy^{-1} \in V_0$. Then there is k with $xy^{-1}a_k \in L$, as above. Also, we have $xy^{-1}a_k \in V_0a_k \subseteq V_k$ and $a_k \in V_0a_k \subseteq V_k$. As also $a_k \in L$, we conclude that $\alpha(xy^{-1}a_k)\alpha(a_k)^{-1} \in U$. Choose any $z \in E_0$. Then $a_k = zz^{-1}a_k$. As ψ is affine and agrees with α on L, we see that

$$\begin{aligned}
\psi(x)\psi(y)^{-1} &= \psi(x)\psi(y)^{-1}\psi(a_k)\psi(a_k)^{-1}\psi(z)\psi(z)^{-1} \\
&= \psi(x)\psi(y)^{-1}\psi(a_k)\big(\psi(z)\psi(z)^{-1}\psi(a_k)\big)^{-1} \\
&= \psi(xy^{-1}a_k)\psi(zz^{-1}a_k)^{-1} = \alpha(xy^{-1}a_k)\alpha(a_k)^{-1} \in U.
\end{aligned}$$

It follows that ψ is (uniformly) continuous. Indeed, if $x \in E_0$ and (x_i) is a net in E_0 converging to x, then $x_i^{-1}x \to e$ in H. Given any neighbourhood U of e in G, pick V_0 as above, and observe that eventually $x_i^{-1}x \in V_0$. Thus

$$\psi(x_i^{-1}x) = \psi(x_i)^{-1}\psi(x) \in U.$$

This shows that $\psi(x_i)^{-1}\psi(x) \to e$ in G so that $\psi(x_i) \to \psi(x)$. □

We are now in a position to complete our argument. We recall the setup from the start of Section 2, so $\Phi\colon A(G) \to B(H)$ is a completely bounded homomorphism, and with G amenable, and there is $\alpha\colon H \to G_\infty$ continuous, which implements Φ. With $Y = \alpha^{-1}(G)$, we additionally know that $\alpha\colon Y \subseteq H \to G$ is piecewise affine.

Proposition 4.4. *With $\alpha\colon Y \subseteq H \to G$ as above, we have that Y is clopen. Furthermore, Y may be written as the disjoint union of sets L_i of the form (3.2), say*

$$L_i = E_0^{(i)} \setminus \bigcup_j E_j^{(i)},$$

with each $E_j^{(i)}$ Borel, and for each i, there is a continuous affine map $\overline{\alpha}_i\colon \overline{E_0^{(i)}} \to G$ which restricts to α on L_i.

Proof. Write Y as the disjoint union of sets of the form (3.2), say $Y = \bigsqcup_i L_i$, and for each i, there is an affine map α_i which agrees with α on L_i. By Proposition 4.3, we know that α_i is continuous, and as α_i is affine, it admits a (unique) continuous extension to the closure of the coset on which it is defined; compare [4, Lemma 1.3 (i)].

We claim that $\overline{L_i} \subseteq Y$. Indeed, given $x \in \overline{L_i}$, there is a net (x_j) in L_i which converges to x, so $\lim_j \alpha_i(x_j) = \overline{\alpha}_i(x)$, and as α_i extends α, also $\lim_j \alpha(x_j) = \overline{\alpha}_i(x)$. As $\alpha\colon H \to G_\infty$ is continuous, we conclude that $\alpha(x) = \overline{\alpha}_i(x) \in G$, and so $x \in Y$ (that is, x is not the point ∞). Notice that we have also shown that α agrees with $\overline{\alpha}_i$ on $\overline{L_i}$. Thus we have that

$$Y = \bigcup_{i=1}^{n} L_i \subseteq \bigcup_{i=1}^{n} \overline{L_i} \subseteq Y,$$

and so we have equality throughout. Hence Y is closed (and also open).

Now consider the graph $\Gamma = \mathcal{G}(\alpha) = \{(y, \alpha(y)) : y \in Y\} \subseteq H \times G$. That α is continuous shows that Γ is closed, and that α is piecewise affine shows that $\Gamma \in \Omega(H \times G)$. By Theorem 3.2, there are subgroups $K_1, \ldots, K_n$ in $\mathcal{R}_2(\{\Gamma\})$ so that $\Gamma \in \mathcal{R}(\{K_1, \ldots, K_n\})$. By Lemma 3.3, we can write $\Gamma = \bigsqcup_i \Gamma_i$, where each Γ_i is of the form (3.2), say $\Gamma_i = F_0^{(i)} \setminus \bigcup_j F_j^{(i)}$, with each $F_j^{(i)}$ a coset of some K_k, and with $F_j^{(i)} \subseteq F_0^{(i)}$ for each j. From Lemma 3.3, we also know that each K_k is of the form $C \cap U$ for some closed set C and some open set U because Γ is closed, and so translates of Γ are closed. From Lemma 3.3 once more, it follows that each member of $\mathcal{R}(\{K_i\})$ is also a finite union of sets of the form "closed intersect open"; in particular, this applies to each $F_j^{(i)}$.

The proof of [4, Lemma 1.2 (iii)] (Lemma 2.1) shows that each $F_0^{(i)} \subseteq H \times G$ is the graph of an affine map, say $\phi_i\colon E_0^{(i)} \to G$, for some coset $E_0^{(i)}$ of H. Then $F_j^{(i)} \subseteq F_0^{(i)}$ is also a graph of the restriction of ϕ_i to a coset, say $E_j^{(i)}$. Thus, in the start of the proof, we could actually take $L_i = E_0^{(i)} \setminus \bigcup_j E_j^{(i)}$ and $\alpha_i = \phi_i$. In particular, each ϕ_i is continuous. Applying Lemma 4.1 to the restriction of ϕ_i to $E_j^{(i)}$ shows that $E_j^{(i)}$ is Borel (as $F_j^{(i)}$ is the graph). □

We now suppose that H is σ-compact. Under this hypothesis, Steinhaus's theorem (see [9]) shows that if $C \subseteq H$ is a coset of non-zero (Haar) measure, then C is open. In the following proof, we use this to show that a finite family of *Borel* cosets, each of which has empty interior, also has union with empty interior. This is exactly the point which goes wrong in the attempted purely topological proof of [4, Lemma 1.3].

Proposition 4.5. *Let H be σ-compact, and continue with the notation of Proposition 4.4. There are $Y_1, \ldots, Y_m$ in the open coset ring of H so that Y is the disjoint union of the Y_i, and for each i, there is a continuous affine map $\alpha_i\colon \mathrm{aff}(Y_i) \to G$ which agrees with α on Y_i.*

Proof. If $E_0^{(i)}$ is open, then as it is a coset, it must also be closed. Reorder so that $E_j^{(i)}$ is open for $1 \leq j \leq m$ and not open for $j > m$. Let $Z_i = E_0^{(i)} \setminus \bigcup_{j=1}^m E_j^{(i)}$. As each $E_j^{(i)}$ is clopen for $j \leq m$, it follows that Z_i is clopen.

For $j > m$, as $E_j^{(i)}$ is not open, it has measure zero, and so also $\bigcup_{j>m} E_j^{(i)}$ has measure zero, and so $\bigcup_{j>m} E_j^{(i)}$ has empty interior. As $L_i \subseteq Z_i$ it follows that $Z_i \setminus L_i \subseteq \bigcup_{j>m} E_j^{(i)}$ so $Z_i \setminus L_i$ has empty interior. As Z_i is open, we have shown that $Z_i \subseteq \overline{L_i}$.

However, Z_i is closed, so as $L_i \subseteq Z_i$ also $\overline{L_i} \subseteq Z_i$. We conclude that $\overline{L_i} = Z_i$ is clopen, and clearly in the open coset ring of H.

Now reorder so that $E_0^{(i)}$ is open for $i \leq m$ and not for $i > m$. For $i > m$, we again have that $E_0^{(i)}$ has measure zero, so also L_i has measure zero, so we again conclude that $\bigcup_{i>m} L_i$ has empty interior. As Y is clopen, and $\bigcup_{i=1}^m \overline{L_i}$ is clopen, it follows that $Y \setminus \bigcup_{i=1}^m \overline{L_i}$ is open. As $\bigcup_{i=1}^m L_i \subseteq \bigcup_{i=1}^m \overline{L_i}$, it follows that $Y \setminus \bigcup_{i=1}^m \overline{L_i} \subseteq Y \setminus \bigcup_{i=1}^m L_i = \bigcup_{i>m} L_i$, and so $Y \setminus \bigcup_{i=1}^m \overline{L_i}$ has empty interior and hence must be empty. So $Y \subseteq \bigcup_{i=1}^m \overline{L_i}$, but then

$$\bigcup_{i=1}^m \overline{L_i} \subseteq \overline{\bigcup_{i=1}^m L_i} \subseteq \overline{Y} = Y,$$

so we conclude that $Y = \bigcup_{i=1}^m \overline{L_i}$.

Finally, we use that α_i extends continuously to $\overline{\alpha}_i$ a continuous affine map; restrict this to $\overline{L_i}$. It seems possible that the $(\overline{L_i})$ are not disjoint, but if we replace $\overline{L_2}$ by $\overline{L_2} \setminus \overline{L_1}$, then we do not leave the open coset ring, and so we can simply adjust to obtain the disjoint family (Y_i) as required. □

We can finally state and prove the main part of [4, Theorem 3.7].

Definition 4.6. Let G, H be locally compact groups. A map $\alpha: Y \subseteq H \to G$ is a *continuous piecewise affine* map when Y is open, and can be written as a disjoint union $Y = \bigsqcup_i Y_i$ with each $Y_i \in \Omega_o(H)$, and for each i, there is an open coset C_i and a continuous affine map $\alpha_i: C_i \to G$, with $Y_i \subseteq C_i$ and α_i agrees with α on Y_i.

Notice that each $Y_i \in \Omega_o(H)$ is also closed, and so also Y is closed; compare also Remark 4.8 below.

Theorem 4.7. *Let $\Phi: A(G) \to B(H)$ be a completely bounded homomorphism, with G amenable. There is a continuous piecewise affine map $\alpha: Y \subseteq H \to G$ with*

$$\Phi(u)(h) = \begin{cases} u(\alpha(h)), & h \in Y, \\ 0, & h \notin Y \end{cases} \qquad (u \in A(G),\ h \in H).$$

Proof. We have already proved this in the σ-compact case. Now let H be an arbitrary locally compact group. With L_i as in Proposition 4.4, we again wish to prove that $Y = \bigcup_{i=1}^m \overline{L_i}$, where $E_0^{(i)}$ is open for $i \leq m$, and not open otherwise.

There is $H_0 \leq H$ an open (and so closed) σ-compact subgroup. Let β be the restriction of α to $Y \cap H_0$. We can apply Proposition 4.5 to β and so conclude that $H_0 \cap Y$ is the union of the sets $\overline{H_0 \cap L_i}$ for those i with $H_0 \cap E_0^{(i)}$ open and non-empty. However, if $E_0^{(i)}$ is open, then also $H_0 \cap E_0^{(i)}$ is open, and if it is empty, there is no harm in considering it in the union. Thus $H_0 \cap Y = \bigcup_{i=1}^m \overline{H_0 \cap L_i}$. This argument would also apply to any translate of Y, equivalently, to any coset of H_0, so we conclude that $sH_0 \cap Y = \bigcup_{i=1}^m \overline{sH_0 \cap L_i}$ for any s. As each sH_0 is clopen, it follows that $Y = \bigcup_{i=1}^m \overline{L_i}$ as required. □

Remark 4.8. On [4, p. 487], $\alpha: Y \subseteq H \to G$ is defined to be "continuous piecewise affine" when α is piecewise affine, and Y is clopen in H. If α is of this form, then we can extend α to a map $\alpha: H \to G_\infty$ by defining $\alpha(y) = \infty$ for $y \notin Y$, and then α will still be continuous because Y is clopen. Then we are in exactly the situation of Proposition 4.4, and so the results above imply that α is a continuous piecewise affine map in our sense.

As such, the use of [4, Lemma 1.3 (ii)] in the proof of the converse of the result above, [4, Proposition 3.1], is also corrected.

The original use of [4, Lemma 1.3 (ii)] was to show that if $\alpha: Y \subseteq G \to H$ is piecewise affine, and continuous, with Y open, then also $\overline{Y}$ is open, and there is a continuous piecewise affine map $\overline{\alpha}: \overline{Y} \to H$ extending α. We have been unable to decide if this result is true or not.

Acknowledgments. We thank Yemon Choi for useful remarks on a preprint of this paper, and the referee for helpful comments which have improved the clarity of the paper.

Bibliography

[1] P. J. Cohen, On homomorphisms of group algebras, *Amer. J. Math.* **82** (1960), 213–226.

[2] P. Eymard, L'algèbre de Fourier d'un groupe localement compact, *Bull. Soc. Math. France* **92** (1964), 181–236.

[3] M. Ilie, On Fourier algebra homomorphisms, *J. Funct. Anal.* **213** (2004), no. 1, 88–110.

[4] M. Ilie and N. Spronk, Completely bounded homomorphisms of the Fourier algebras, *J. Funct. Anal.* **225** (2005), no. 2, 480–499.

[5] B. H. Neumann, Groups covered by permutable subsets, *J. Lond. Math. Soc.* **29** (1954), 236–248.

[6] M. Noy, Graphs, in: *Handbook of Enumerative Combinatorics*, Discrete Math. Appl. (Boca Raton), CRC Press, Boca Raton (2015), 397–436.

[7] H. L. Pham, Contractive homomorphisms of the Fourier algebras, *Bull. Lond. Math. Soc.* **42** (2010), no. 5, 937–947.

[8] W. Rudin, *Fourier Analysis on Groups*, Wiley Classics Lib., John Wiley & Sons, New York, 1990.

[9] K. Stromberg, An elementary proof of Steinhaus's theorem, *Proc. Amer. Math. Soc.* **36** (1972), 308.

[10] YCor https://mathoverflow.net/users/14094/ycor, Empty interior of union of cosets?, https://mathoverflow.net/q/351846.

Received March 10, 2021; revised September 14, 2021

Author information

Corresponding author:
Matthew Daws, Jeremiah Horrocks Institute, University of Central Lancashire,
Preston, PR1 2HE, United Kingdom.
E-mail: matt.daws@cantab.net