

## Central Lancashire Online Knowledge (CLoK)

Title	Probing the shape and history of the Milky Way halo with orbital spectral analysis
Type	Article
URL	<a href="https://cllok.uclan.ac.uk/id/eprint/4437/">https://cllok.uclan.ac.uk/id/eprint/4437/</a>
DOI	<a href="https://doi.org/10.1111/j.1365-2966.2011.19853.x">https://doi.org/10.1111/j.1365-2966.2011.19853.x</a>
Date	2011
Citation	Valluri, Monica, Debattista, Victor P, Quinn, Thomas R., Roškar, Rok and Wadsley, James (2011) Probing the shape and history of the Milky Way halo with orbital spectral analysis. Monthly Notices of the Royal Astronomical Society, 419 (3). pp. 1951-1969. ISSN 00358711
Creators	Valluri, Monica, Debattista, Victor P, Quinn, Thomas R., Roškar, Rok and Wadsley, James

It is advisable to refer to the publisher's version if you intend to cite from the work.  
<https://doi.org/10.1111/j.1365-2966.2011.19853.x>

For information about Research at UCLan please go to <http://www.uclan.ac.uk/research/>

All outputs in CLoK are protected by Intellectual Property Rights law, including Copyright law. Copyright, IPR and Moral Rights for the works on this site are retained by the individual authors and/or other copyright owners. Terms and conditions for use of this material are defined in the <http://cllok.uclan.ac.uk/policies/>

# Probing the shape and history of the Milky Way halo with orbital spectral analysis

Monica Valluri,<sup>1★</sup> Victor P. Debattista,<sup>2★†</sup> Thomas R. Quinn,<sup>3</sup> Rok Roškar<sup>4</sup>  
and James Wadsley<sup>5</sup>

<sup>1</sup>*Department of Astronomy, University of Michigan, Ann Arbor, MI 48109, USA*

<sup>2</sup>*Jeremiah Horrocks Institute, University of Central Lancashire, Preston PR1 2HE*

<sup>3</sup>*Astronomy Department, University of Washington, Box 351580, Seattle, WA 98195-1580, USA*

<sup>4</sup>*Department of Theoretical Physics, University of Zürich, Winterthurerstrasse 190, CH-8057 Zürich, Switzerland*

<sup>5</sup>*Department of Physics and Astronomy, McMaster University, Hamilton, Ontario L8S 4M1, Canada*

Accepted 2011 September 17. Received 2011 September 7; in original form 2011 June 23

## ABSTRACT

Accurate phase-space coordinates (three components of position and velocity) of individual stars are rapidly becoming available with current and future resolved star surveys. These data will enable the computation of the full three-dimensional orbits of tens of thousands of stars in the Milky Way's stellar halo. We demonstrate that the analysis of stellar halo orbits in frequency space can be used to construct a 'frequency map' which provides a highly compact, yet intuitively informative way to represent the six-dimensional halo phase-space distribution function. This representation readily reveals the most important major orbit families in the halo, and the relative abundances of the different orbit families, which in turn reflect the shape and orientation of the dark matter halo relative to the disc. We demonstrate the value of frequency space orbit analysis by applying the method to halo orbits in a series of controlled simulations of disc galaxies. We show that the disc influences the shape of the inner halo making it nearly oblate, but the outer halo remains largely unaffected. Since the shape of the halo varies with radius, the frequency map provides a more versatile way to identify major and minor orbit families than traditional orbit classification schemes. Although the shape of the halo varies with radius, frequency maps of local samples of halo orbits confined to the inner halo contain most of the information about the global shape of the halo and its major orbit families. Frequency maps show that adiabatic growth of a disc traps halo orbits in numerous resonant orbit families (i.e. having commensurable frequencies). The locations and strengths of these resonant families are determined by both the global shape of the halo and its stellar distribution function. If a good estimate of the Galactic potential in the inner halo (within  $\sim 50$  kpc) is available, the appearance of strong, stable resonances in frequency maps of halo orbits will allow us to determine the degree of resonant trapping induced by the disc potential. We show that if the Galactic potential is not known exactly, a measure of the diffusion rate of a large sample of  $\sim 10^4$  halo orbits can help distinguish between the true potential and an incorrect potential. The orbital spectral analysis methods described in this paper provide a strong complementarity to existing methods for constraining the potential of the Milky Way halo and its stellar distribution function.

**Key words:** methods: numerical – Galaxy: evolution – Galaxy: halo – Galaxy: kinematics and dynamics – Galaxy: structure – dark matter.

## 1 INTRODUCTION

Over the last decade a coherent picture of the formation of the stellar halo of the Milky Way (MW) has begun to emerge from both observational surveys of the Galaxy and cosmologically motivated

★E-mail: mvalluri@umich.edu (MV); vpdebattista@uclan.ac.uk (VPD)

†RCUK Fellow.

simulations. In this ‘concordance cosmological model’ galaxies are embedded in dark matter haloes and form via the merger of proto-galactic fragments. The fragments consist of dark matter, gas and the first generations of stars which formed in high-density peaks in the early Universe. An extended period of ‘late infall’ including the accretion and tidal shredding of numerous dwarf satellite galaxies (Bullock & Johnston 2005; De Lucia & Helmi 2008; Font et al. 2008; Johnston et al. 2008; Cooper et al. 2010) has continued to build up the halo even at the present time. In contributing to the growth of disc galaxies, gas accretion competes with accretions and mergers of satellites which tend to thicken and disrupt the disc. Recent resolved-star all-sky surveys such as Sloan Digital Sky Survey-Sloan Extension for Galactic Understanding and Exploration (SDSS-SEGUE; Yanny et al. 2009) and RAdial Velocity Experiment (RAVE; Steinmetz et al. 2006) have revealed that current observations of the stellar halo are broadly consistent with the  $\Lambda$ CDM dark matter ( $\Lambda$ CDM) galaxy formation paradigm. For example the detection in SDSS data of numerous substructures in the MW halo in the form of tidal streams (Newberg et al. 2002; Yanny et al. 2003; Belokurov et al. 2006), the measurement of the degree of clumpiness in the distribution of stars in the stellar halo (Bell, Zucker & co authors 2008) and the discovery of numerous ultra-faint dwarf spheroidal galaxies (Willman et al. 2005; Zucker et al. 2006; Belokurov et al. 2007) have all bolstered evidence that the MW’s stellar halo was produced in the manner consistent with  $\Lambda$ CDM.

Although this picture of galaxy formation has been very successful, the availability of phase-space coordinates for tens of thousands of halo stars from resolved star surveys have revealed the inadequacies, and inherent degeneracies, of current analysis tools. For instance, phase-space data are currently compared with simulations using simple measures such as orbital eccentricity (Sales et al. 2009), and not only yield degenerate results (Dierickx et al. 2010; Wilson et al. 2011), but also depend on assumptions regarding the potential of the Galaxy. Although gas accretion probably dominates in disc galaxies it is unclear from the current data whether the MW’s inner stellar halo, and thick disc formed purely *in situ* (Schönrich & Binney 2009; Loebman et al. 2010) or whether a pre-existing thin disc was heated (Villalobos & Helmi 2008; Kazantzidis et al. 2008) or if they were entirely created by satellite accretion (Abadi et al. 2003; Yoachim & Dalcanton 2008). There is also evidence that the inner stellar halo may be smoother (less *rms* density variation) and more metal rich than the outer halo (Helmi et al. 2011). In addition, a recent analysis of the full space motions of  $\sim 17\,000$  halo stars from the SDSS-SEGUE calibration sample (Carollo et al. 2007, 2010; Beers et al. 2011) suggests that the Galactic halo consists of two overlapping components: an inner halo, which is rotating in the same direction as the disc, and an outer halo, with a small retrograde motion. In addition, the two components have different density profiles, stellar orbits and metallicities (however see Schönrich, Asplund & Casagrande 2011, for a different view).

Current analysis tools rely on comparing the average distributions of stars in the halo with those arising from simulations but are not powerful enough to compare the full six-dimensional phase-space distribution functions (DF) of  $10^5$ – $10^6$  halo stars and the information on their metallicities and ages that will become available following the launch of *Gaia* (Perryman et al. 2001). It is impossible for current analyses to rule out the possibility that a significant fraction of nearby halo stars are not formed in our own potential, and have been heated up into a very thick spheroidal inner halo. Addressing questions of this kind are of key importance when trying to constrain galaxy formation models, and in uncovering the formation history

of the MW, and can only be achieved with new techniques that can quantitatively compare the self-consistent chemodynamical and phase-space DF of the stellar halo with those from models.

In this paper we show that frequency analysis tools can be applied to orbits of halo stars to uncover the phase-space DF of the entire stellar halo. These methods can be applied to both the real DF of MW halo orbits and to those from  $N$ -body simulations, permitting detailed quantitative comparisons of the orbit families in the DFs. We also show that the halo orbit DF (represented by a frequency map) reflects the global shape of the dark matter halo and its orientation relative to the disc.

One of the robust predictions of the  $\Lambda$ CDM cosmological paradigm is that the dark matter haloes of galaxies like the MW are triaxial (Dubinski & Carlberg 1991; Jing & Suto 2002; Bailin & Steinmetz 2005; Allgood et al. 2006). Orbits of particles in triaxial potentials are very different from those in spherical and axisymmetric potentials. For instance, none of the orbits conserves any component of angular momentum. The fraction of such ‘triaxial orbits’ in a potential is a strong indicator of the global shape of the potential. If haloes are elongated in ways that reflect their accretion history and their orientation relative to large-scale filaments (Helmi et al. 2011), determining the shape and the orbital populations could give us important clues about the formation of the MW.

Recent simulations have shown that when baryons cool and condense at the centres of triaxial dark matter haloes, the haloes become more axisymmetric especially at small radii (Dubinski & Carlberg 1991; Kazantzidis et al. 2004; Debattista et al. 2008, hereafter D08; Kazantzidis, Abadi & Navarro 2010; Tissera et al. 2010). Two recent studies (Valluri et al. 2010, hereafter V10; D08) analysed the orbital properties of halo particles in a series of  $N$ -body simulations in which baryonic components were grown adiabatically in triaxial and prolate haloes. V10 showed that although the inner one-third of the halo becomes nearly oblate following the growth of a baryonic component, only a fraction of the orbits change their true orbital characteristics. When the baryonic component is a disc galaxy, most of the halo orbits retain a memory of their orbital actions. Since the actions are adiabatic invariants, even nearly oblate haloes can have a significant fraction of box orbits and long-axis tubes.

This ability of orbits to retain a memory of their initial conditions has been previously noted in several numerical studies: when the potential is changing fairly violently e.g. during a major merger (Valluri et al. 2007) or during the tidal disruption of a satellite in the field of a larger galaxy (Helmi & de Zeeuw 2000), stellar orbits largely conserve their integrals of motion. In the latter case Helmi & de Zeeuw (2000) showed that the orbital actions can be used to recover the relics of the tidal disruption of a dwarf satellite by the MW potential. Recently, a number of studies (McMillan & Binney 2008; Gómez & Helmi 2010; Gómez et al. 2010) have shown that the orbital frequencies  $\Omega_R$  (radial oscillation or epicyclic frequency) and  $\Omega_\phi$  (tangential oscillation or rotation frequency) of stars belonging to a single satellite are tightly correlated and that this correlation remains strong long after the remnant has become completely well mixed in configuration (physical) space. Consequently, correlations between orbital frequencies can be used to identify stars that belong to individual accretion events, and simultaneously determine the true galactic potential and the time since each satellite galaxy was disrupted (McMillan & Binney 2008; Gómez & Helmi 2010).

Following the method outlined by Carpintero & Aguilar (1998), V10 (also Deibel, Valluri & Merritt 2011) used the oscillation frequencies of orbits of particles in simulated haloes to classify orbits into major families. V10 also used orbital frequencies to quantify the shapes of orbits (and to relate the orbital shapes to the shapes

of the haloes), to identify orbits which are chaotic and to identify important resonances (regions of phase space occupied by orbits with commensurable orbital frequencies). In this paper we show that the *relative distribution of orbital types and the identification of important resonances* populated by halo orbits strongly reflect the orientation of the triaxial halo relative to the galactic disc. We will show that the discovery of large numbers of halo stellar orbits trapped in resonances could put constraints on the form of the Galactic potential and the DF of the stellar halo.

Why are resonances important? When a time-dependent force acts on a system like a galaxy, long-lived resonant interactions play an important role in the evolution of the system and leave imprints in its phase-space structure. The identification of stars trapped in resonances can put constraints on the potential and its components (e.g. the bar and spiral arms). Resonant interactions between individual orbits and a changing potential influence time-dependent (secular) evolution: e.g. it can cause stars to ‘levitate’ to form a thick disc (Sridhar & Touma 1996), it may result in the formation of polar rings or counter-rotating discs (Tremaine & Yu 2000), and it can cause resonant shocking (or torquing) of stars in satellites as they are disrupted in dark matter haloes (Choi, Weinberg & Katz 2009). ‘Capture into resonance’ is studied extensively in the context of planetary systems, where a migrating planet can capture planets or planetesimals into mean motion resonances (Malhotra 1993; Yu & Tremaine 2001). In the planetary dynamics literature it has been shown that resonant trapping of planetesimals occurs in slowly varying potentials, but can be prevented when the drift or migration rate is sufficiently high (i.e. non-adiabatic) (e.g. Quillen 2006). The identification of a significant fraction of resonantly trapped halo orbits could therefore provide clues about the way in which the halo potential has changed over time.

Laskar’s frequency analysis method is particularly good at identifying resonances (Robutel & Laskar 2001). We show that when the method is applied to orbits in a self-consistent DF, the method permits easy identification of the major orbit families and an assessment of the relative importance of each family to the phase-space DF.

This paper is organized as follows. Section 2 describes the simulations analysed in this paper and briefly describes the frequency analysis of orbits. Section 3 presents the results of our analysis of adiabatic simulations of isolated galactic potentials consisting of both axisymmetric and triaxial dark matter haloes with particle discs of various orientations (taken mostly from D08). We also discuss the effects of disc galaxies which form self-consistently from hot halo gas in a spherical or prolate halo. These simulations include the hydrodynamical effects of gas cooling, star formation and supernova feedback (Stinson et al. 2006; Roškar et al. 2008). In Section 4 we show how the mean orbital diffusion rates of a large

ensemble of orbits selected from a distribution function can be used to assess how much it deviates from self-consistent equilibrium and discuss how this measure is affected when the assumed Galactic potential is incorrect. In Section 5 we summarize our results, and discuss their implications for future large data sets which will obtain the six-dimensional orbits of stars in the MW halo.

## 2 SIMULATIONS AND NUMERICAL METHODS

### 2.1 Simulations

We analysed two types of controlled simulations: (a)  $N$ -body simulations in which an exponential, thin stellar disc (consisting of collisionless particles) was grown adiabatically inside an isolated halo; (b)  $N$ -body+smoothed particle hydrodynamic (SPH) simulations of the formation of stellar discs from initially hot gas distributed inside a spherical or prolate halo and allowed to cool and form a disc of gas in which stars form. In general, we assume that the potential of the galaxy model is perfectly known, and we characterize the DF, its orbital properties and its dependence on the radial variation of the shape of the halo and its orientation relative to that of the disc. In practice, the potential and the DF of the halo need to be determined simultaneously, or (in the absence of adequate kinematical constraints) the potential alone will be determined, within some radius. We therefore ran a few additional simulations, designed to determine if it is possible to constrain the potential from the orbital properties of an ensemble of halo orbits.

In the controlled simulations presented in this paper, initially spherical isotropic NFW (Navarro, Frenk & White 1996) haloes were generated via Eddington’s formula (Binney & Tremaine 2008, section 4.3.1) with each halo composed of two mass species arranged on shells. The inner shell has less massive particles than the outer one, which allows for higher mass resolution at small radii. Most of the dark matter particles in the inner part of the halo have masses of  $10^6 M_\odot$ . Prolate and triaxial haloes (consisting of  $4 \times 10^6$  particles) were generated via mergers of the spherical NFW haloes (see D08 for details).

In the simulations, discs of particles were grown (starting from nearly zero initial mass) adiabatically and linearly on a time-scale  $t_g$  inside a dark matter halo (see Tab. 1 for details of parameters of the simulations). Discs were grown in a spherical halo (model SNFWD) and in triaxial haloes with the disc plane oriented in various ways relative to the principle axes of the halo: (a) perpendicular to the short axis (model SA1), (b) perpendicular to the long axis (model LA1), (c) perpendicular to the intermediate axis (model IA1) and (d) tilted at an angle of  $30^\circ$  to the  $xy$  plane of the triaxial halo, by

**Table 1.** The collisionless simulations in this paper.  $M_b$  is the mass in baryonic disc and  $f_b$  is the baryonic mass fraction.  $t_g$  is the time during which the baryonic disc is grown. In all the models the exponential disc has a radial scalelength of 3 kpc. The last column contains the references where the simulation was first reported.

Run name	$r_{200}$ (kpc)	$M_{200}$ ( $10^{12} M_\odot$ )	$M_b$ ( $10^{11} M_\odot$ )	$f_b$	$t_g$ (Gyr)	Run description	Reference
SNFWD	85	0.66	0.66	0.1	5	Spherical halo+stellar disc	This paper
SA1	215	4.5	1.75	0.039	5	Triaxial halo+short-axis stellar disc	D08
LA1	215	4.5	1.75	0.039	5	Triaxial halo+long-axis stellar disc	D08
IA1	215	4.5	1.75	0.039	5	Triaxial halo+intermediate (y) axis stellar disc	D08
TA1	215	4.5	1.75	0.039	5	Triaxial halo+tilted stellar disc	D08
SA2	215	4.5	0.52	0.012	1.5	Triaxial halo+barred stellar disc	D08 (there labelled BA1)



rotating it about the  $y$ -axis (model TA1).<sup>1</sup> D08 showed that in all of these models, the shape of the halo within the inner 1/3 of the virial radius becomes nearly (but not exactly) oblate following the growth of the disc, with the short axis of the oblate part of the halo co-aligned with the spin axis of the disc. In most of the simulations the disc particles remain stationary, hence the disc is rigid throughout. In one case (model SA2), a ‘light’ disc of particles was made ‘live’ after the disc had grown to its final mass. This disc subsequently formed a bar which persisted for  $\sim 10$  Gyr before dissolving because of the triaxial halo (Berentzen & Shlosman 2006). Additional details may be found in D08. All the collisionless simulations were evolved with PKDGRAV, an efficient, multi-stepping, parallel tree code (Stadel 2001). Dark matter particles had a softening parameter  $\epsilon = 0.1$  kpc, and that of stars was in the range  $\epsilon = 60\text{--}100$  pc.

Two of our controlled simulations contain a baryonic (gas+star) disc, which forms self-consistently from hot gas in a spherical (model SNFWgs) or prolate (model SBgs) halo. The baryonic component is 10 per cent of the total mass and initially has the same density distribution as the dark matter particles. The halo and gas particles are given an initial specific angular momentum  $j$ , determined by overall cosmological spin parameter  $\lambda = (j/G)(|E|/M^3)^{1/2} = 0.039$ , which is motivated by cosmological  $N$ -body experiments (Bullock et al. 2001). Both the spherical halo and the progenitor haloes that were merged to produce a prolate halo had the same angular momentum parameter  $\lambda$ . Each component is modelled with  $10^6$  particles, with the dark matter particles of mass  $10^6 M_\odot$ , and gas particles having an initial mass of  $10^5 M_\odot$ . The gas particles are allowed to cool and form stars of typical mass around  $3 \times 10^4 M_\odot$  following the prescription in Stinson et al. (2006). The net angular momentum allows the gas to form a disc as it cools, resulting in a stellar disc as star formation occurs. The simulation closely follows that described in Roškar et al. (2008) and is evolved with the parallel  $N$ -body+SPH code GASOLINE (Wadsley, Stadel & Quinn 2004) for 10 Gyr.

Although controlled simulations are useful for testing the effects of discs with different orientations on the halo DF, these models are not fully realistic depictions of how disc galaxies form. In the current hierarchical structure formation paradigm, disc galaxies probably experienced several gas-rich mergers, at least in their early history, and continue to accrete small satellites today which add to the stellar halo. We defer the study of stellar haloes from cosmological simulations drawn from the McMaster Unbiased Galaxy Simulations (MUGS) project (Stinson et al. 2010) to a future paper (Valluri et al., in preparation).

## 2.2 Selecting the halo orbit samples

In each of the simulations we selected  $1\text{--}2 \times 10^4$  dark matter particles. The particles were either randomly distributed within some spherical volume of radius  $r_g$  centred on the model’s galactic centre, or within a region of radius  $R_s$  from the location of the ‘Sun’ (which was assumed to be at 8 kpc from the Galactic centre). Since most of the potentials we studied are non-axisymmetric, the azimuthal location of the ‘Sun’ in the equatorial plane relative to the major axis of the triaxial halo is an additional parameter. Rather than choosing a specific (arbitrary) angle in azimuth for the solar location, we select particles within a ‘torus’ of width  $R_s$  and with radius 8 kpc. When we selected subsamples of the torus region we found that the results

did not depend much on the precise azimuthal location of the ‘Sun’, so long as the radial region sampled was 10 kpc in size, or larger.

Following V10 we studied orbits of halo particles after the disc had grown to its final mass and the halo had relaxed to a new equilibrium. The orbits were integrated for 50 Gyr in a frozen potential corresponding to the new equilibrium potential given by the full mass distribution of the simulation (dark matter and baryons) using an integration scheme based on the PKDGRAV tree. The 50 Gyr integrations are used only because they yield highly accurate frequencies for the halo orbits with the longest periods, and should not be construed to imply that this is a physically meaningful time period. (In a similar way, it is possible to integrate orbits of MW halo stars in a fixed potential for equally long times in order to characterize the nature of their *current* orbit.) None of the controlled simulations includes a stellar halo, so we assume that the dark matter halo orbits can be used to represent the orbits of particles in the stellar halo. While this is clearly not an ideal assumption, many studies of the stellar halo use prescriptions to ‘tag’ the most tightly bound dark matter particles in dwarf satellites as ‘star particles’ (e.g. Bullock & Johnston 2005; Cooper et al. 2010). Elsewhere we analyse a fully cosmological hydrodynamical simulation of a disc galaxy from the MUGS project (Stinson et al. 2010) and show that in the inner halo (a region on which we focus) star particles and dark matter particles have similar orbital properties, justifying this assumption (Valluri et al., in preparation).

## 2.3 Laskar frequency mapping

In three-dimensional Hamiltonian potentials, the phase-space structure of regular orbits can be described by three actions  $J_\alpha$  ( $\alpha = 1, 2, 3$ ) and three angle variables  $\theta_\alpha$ , which constitute a canonically conjugate coordinate system. The actions are integrals of motion and are conserved along the orbit, while the angles increase linearly with time. The angle variables at any time  $t$  are given by  $\theta_\alpha(t) = \theta_\alpha(0) + \Omega_\alpha t$ , where the  $\Omega_\alpha$  are called ‘fundamental frequencies’. The actions are adiabatic invariants and consequently remain constant as the potential of the system changes adiabatically. Regular orbits in three-dimensional potentials can therefore be thought of as occupying the surfaces of three-dimensional tori, with the size of each torus characterized by the actions  $J_\alpha$ , while the angle variables  $\theta_\alpha(t)$  represent the traversal of the orbit over the surface of the torus in each dimension.

Since the angle-action coordinates are related to the classical spatial coordinates and momenta via a coordinate transformation, it can be shown that traditional space and velocity coordinates can be represented by a time series of the form:  $x(t) = \sum A_k e^{i\omega_k t}$ , and similarly for other phase-space coordinates e.g.  $y(t)$ ,  $v_x(t)$  (where the sum is over all terms in the spectrum). A Fourier transform of such a time series will yield the spectrum of orbital frequencies  $\omega_k$  and associated amplitudes  $A_k$ , that govern the motion of the orbit (Binney & Spergel 1982, 1984; Binney & Tremaine 2008). For most regular orbits only three of the frequencies in the spectrum (of a given orbit) are linearly independent (i.e. all other frequencies  $\omega_k$  can be expressed as  $\omega_k = l_k \Omega_1 + m_k \Omega_2 + n_k \Omega_3$ , where  $l_k, m_k, n_k$  are integers).  $\Omega_1, \Omega_2, \Omega_3$  are nothing other than the ‘fundamental frequencies’ described above and the associated three orbital actions  $J_1, J_2, J_3$  can be computed from the amplitudes  $A_k$ .

If all orbits in the potential are regular and the DF can be written as a continuous function of global actions  $J_1, J_2, J_3$  e.g.  $f(J_1, J_2, J_3)$  then the corresponding fundamental frequencies and the associated angle variables  $\theta_1, \theta_2, \theta_3$  vary in a continuous manner across phase space. However, it is only possible to write the DF as a function of

<sup>1</sup> Note that  $x, y, z$  are defined to be the long, intermediate and short axes, respectively, of the triaxial halo.

global actions for special cases e.g. strictly axisymmetric potentials or separable triaxial (Stäckel) potentials. If analytic global actions are known then one can simply use surfaces of section to map the phase-space structure and to identify resonant orbits and chaotic orbits (BT08). In the study of three-dimensional orbits in realistic potentials analytic integrals of motion other than the Hamiltonian are rarely available. If we compute the orbital frequency spectrum of an orbit in an arbitrary coordinate system, e.g. in Cartesian or cylindrical coordinates, the resulting orbital frequencies  $\Omega_x$ ,  $\Omega_y$ ,  $\Omega_z$  are in no way ‘fundamental’ to the nature of the potential. However, as shown by Laskar (1990) any canonically conjugate pair of variables can be combined to obtain a frequency spectrum, when there are no global actions. These frequencies are still referred to as ‘fundamental frequencies’ since for any regular orbit all the components of the frequency spectrum are linear integer combinations of the three fundamental e.g.  $(\Omega_x, \Omega_y, \Omega_z)$  or  $(\Omega_R, \Omega_\phi, \Omega_z)$ , depending on the coordinate system selected.

Laskar (1990, 1993) developed a very accurate numerical technique ‘Numerical Analysis of Fundamental Frequencies’ (NAFF) to recover frequencies in completely general potentials. We use an implementation of this algorithm due to Valluri & Merritt (1998) which was adapted for application to orbits in  $N$ -body potentials by V10.

The NAFF algorithm recovers orbital frequencies from three complex time series consisting of pairs of phase-space variables. For triaxial potentials we use a Cartesian coordinate system centred on the centre of the galaxy and oriented such that  $x$ ,  $y$ ,  $z$  correspond to the major (long), intermediate and minor (short) axes of the potential, respectively. In the Cartesian coordinates the Fourier analysis is performed on three time series of the form  $f_\alpha(t) = \alpha(t) + i\nu_\alpha(t)$  (where  $\alpha = x, y, z$ ).

For potentials which are axisymmetric or nearly so, most of the orbits are tubes which circulate about the symmetry axis. In such potentials it is preferable to work in cylindrical polar coordinates. We transform from the planar coordinates  $x, y, v_x, v_y$  to plane polar coordinates  $R, v_R, \phi, \Theta$ , where  $R = \sqrt{x^2 + y^2}$ , azimuthal angle  $\phi = \arctan(x/y)$ ,  $v_R = (xv_x + yv_y)/R$  and  $\Theta = xv_y - yv_x$ .  $R, v_R$  are the canonically conjugate radial coordinate and momentum and hence can be used to define a complex time series  $f_R(t) = R(t) + iv_R(t)$ . However, since  $\phi$  and  $\Theta$  are the angular coordinate and momentum (and not linear coordinate and momentum) this pair cannot be used to construct the complex time series used by the frequency analysis method. Following Papaphilippou & Laskar (1996) we use Poincaré’s symplectic polar variables  $\sqrt{2\Theta} \cos \phi$  and  $\sqrt{2\Theta} \sin \phi$  to define the function  $f_\phi = \sqrt{2\Theta}(\cos \phi + i \sin \phi)$ . For motion perpendicular to the equatorial plane we use the complex series  $f_z(t) = z(t) + iv_z(t)$ .

V10 demonstrate that fundamental frequencies of orbits in a self-consistent DF can be used to construct a ‘frequency map’ which gives a picture of the phase-space structure based on its orbital content. A frequency map of phase space is obtained by plotting the *ratios* of fundamental frequencies (e.g. in Cartesian coordinates:  $\Omega_x/\Omega_z$  versus  $\Omega_y/\Omega_z$ ) for a very large number of orbits.

As we noted above the coordinate system selected to integrate the orbits and compute the frequencies is determined more by convenience and convention than any fundamental property of the orbital frequencies obtained. None the less we will see that for triaxial systems, the choice of a Cartesian coordinate system aligned such that the global principal axes of the model coincide with the coordinate system results in different orbit families appearing in distinct groups or lines on a frequency map. The use of an appropriate coordinate system also allows one to identify truly resonant orbit families.

Resonant orbits are regular orbits that have fewer than three linearly independent fundamental frequencies which are related via integer linear equation such as:  $l\Omega_1 + m\Omega_2 + n\Omega_3 = 0$ , where  $(l, m, n)$  are small integers. A frequency map can be used to easily identify the most important resonant orbit families, since such orbits populate straight lines on such a map. The strength (or importance) of a resonance can be assessed from the number of orbits associated with a particular resonance.

Merritt & Valluri (1999) showed that perfectly resonant orbits in three-dimensional potentials have two non-zero fundamental frequencies and occupy thin two-dimensional surfaces (generally multiply connected), in configuration (physical) space. They are surrounded by a resonance region consisting of orbits which share the oscillation frequencies of the perfectly thin resonant parent, but have a third non-zero frequency, which is small but increases as the orbit deviates from its resonant parent. Most of these slightly non-resonant orbits also appear along the resonance lines in the frequency map. Unstable resonances appear as blank lines or blank spaces on the frequency map.

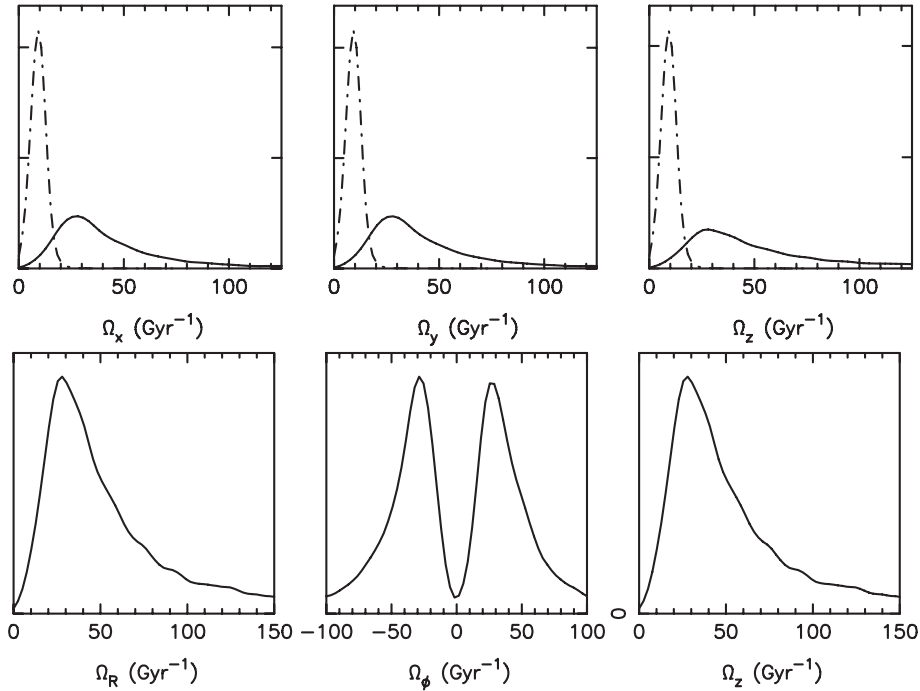
Laskar, Froeschlé & Celletti (1992) also showed that since the frequencies of regular orbits can be recovered with very high accuracy, chaotic orbits can be easily identified, since their frequencies do not remain constant but drift when computed over two adjacent time intervals. They showed that the rate at which the orbits diffuse in frequency space is correlated with their degree of stochasticity. V10 showed that this way of measuring stochasticity was particularly useful in  $N$ -body potentials (and superior to the better known ‘Lyapunov exponent’) since it is able to distinguish between diffusion due to micro-chaos that arises due to discreteness noise (Kandrup & Sideris 2003) and genuinely irregular behaviour. We refer the reader to V10 for more details. For each orbit we divide the integration time of 50 Gyr into two consecutive segments ( $t_1$  and  $t_2$ ) and use NAFF to compute the fundamental frequencies  $\Omega_\alpha(t_1)$ ,  $\Omega_\alpha(t_2)$ . The ‘diffusion’ rate for each frequency component is then computed as

$$\log(\Delta f_\alpha) = \log \left| \frac{\Omega_\alpha(t_1) - \Omega_\alpha(t_2)}{\Omega_\alpha(t_1)} \right|. \quad (1)$$

We define the diffusion rate for an orbit,  $\log(\Delta f)$  (logarithm to base 10), to be the value associated with the frequency component  $\Omega_\alpha$  with the highest amplitude  $A_\alpha$  measured over the entire time interval  $(t_1 + t_2)$ .<sup>2</sup> The larger the value of the diffusion rate  $\log(\Delta f)$ , the more chaotic the orbit. We use the diffusion rate to distinguish between regular and chaotic orbits, and to distinguish weakly chaotic orbits from strongly chaotic orbits. It is important to note that for most systems there is a continuous and nearly Gaussian distribution of diffusion rates (V10).

Laskar’s method recovers orbital frequencies of regular orbits with very high accuracy in  $\sim 20$ – $30$  orbital periods making this method particularly valuable for studying Galactic halo stars. We integrated all orbits for 50 Gyr, but only present results for those with orbital periods shorter than 2.5 Gyr (i.e. those that execute more than 20 orbital periods). All frequencies in this paper are reported in units of  $\text{Gyr}^{-1}$ .

<sup>2</sup> Note that this definition of ‘diffusion rate’ differs slightly from V10 who used the value associated with the largest fundamental frequency. The new definition was found to more accurately identify chaotic orbits and yields a lower rate of misclassification of regular orbits as chaotic.



**Figure 1.** Histograms of orbital frequencies of  $10^4$  halo particles with  $r_g < 200$  kpc. The vertical axes have arbitrary scales. Top row (L–R): histograms of  $\Omega_x$ ,  $\Omega_y$ ,  $\Omega_z$  in a spherical isotropic NFW halo (dot–dashed curves), and after a thin collisionless stellar disc was grown adiabatically (solid curves); bottom row (L–R):  $\Omega_R$ ,  $\Omega_\phi$  and  $\Omega_z$  for the case when a disc is present. The distribution in  $\Omega_\phi$  is bimodal because the original halo had a spherical isotropic DF.

### 3 RESULTS OF CONTROLLED SIMULATIONS

#### 3.1 Nearly oblate axisymmetric haloes

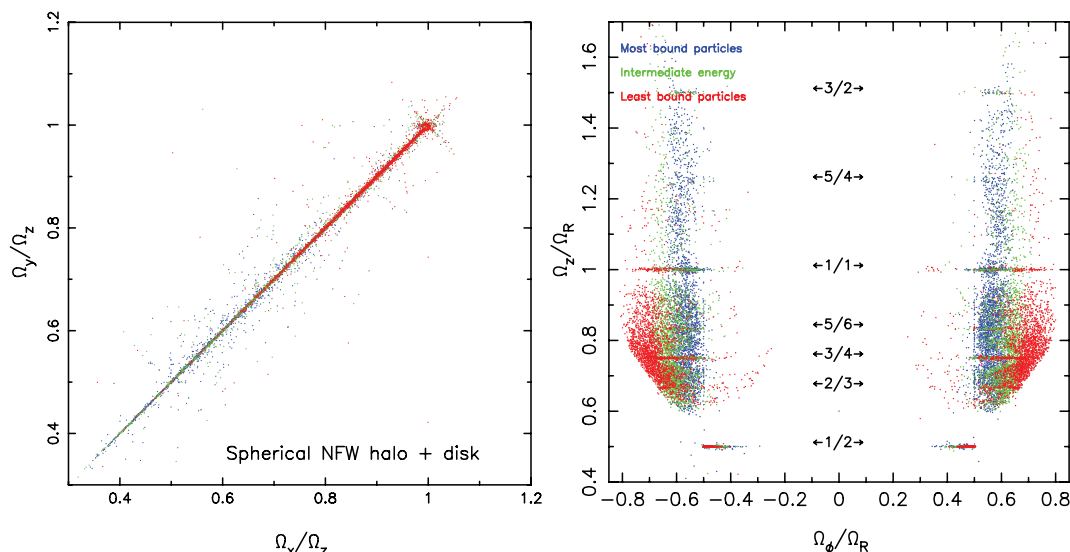
In oblate axisymmetric potentials most orbits are short-axis tubes, hence orbits are best studied in cylindrical coordinates. We begin with the study of the simplest halo DF: a spherical NFW halo in which a stellar disc was grown adiabatically (model SNFWD in Table 1). Fig. 1 shows histograms of frequencies for  $2 \times 10^4$  halo particles with initial  $r_g < 200$  kpc, in Cartesian coordinates  $\Omega_x$ ,  $\Omega_y$ ,  $\Omega_z$  (top panels) and in cylindrical coordinates about the  $z$ -axis  $\Omega_R$ ,  $\Omega_\phi$ ,  $\Omega_z$  (bottom panels).<sup>3</sup> The initial DF was spherical and isotropic and hence the frequency distributions in the three Cartesian directions are identical (dot–dashed curves). The growth of an axisymmetric disc (rotating about the  $z$ -axis) increases the depth of the potential and makes the halo flatter in the inner regions. This results in an increase in all the orbital frequencies, but axisymmetry implies identical distributions for  $\Omega_x$  and  $\Omega_y$ . Since the disc potential is significantly flatter in the  $z$  direction, there is a greater increase in  $\Omega_z$  for particles that lie closer to the centre of potential (highest values of  $\Omega_z$ ), accounting for the slight increase in the weight of the high-frequency tail of the  $\Omega_z$  distribution. In cylindrical coordinates the frequency  $\Omega_\phi$  (bottom row, middle column) describes the motion in the azimuthal direction and is either positive or negative depending on whether the orbit rotates counter-clockwise or clockwise, about the  $z$ -axis. Since the halo was set up to be unrotating,  $\Omega_\phi$  values are symmetrically distributed about zero.

<sup>3</sup> Note that the frequency  $\Omega_R$  is measured relative to the centre of the cylindrical coordinate system and not relative to a circular orbit as in the case of the epicyclic frequency for disc stars.

Fig. 2 shows frequency maps for the same  $2 \times 10^4$  halo particles in Cartesian coordinates (left) and in cylindrical coordinates (right). Each particle is represented by a single point whose location is determined by the ratio of the fundamental frequencies in Cartesian coordinates ( $\Omega_x/\Omega_z$  versus  $\Omega_y/\Omega_z$ ) or cylindrical coordinates ( $\Omega_z/\Omega_R$  versus  $\Omega_\phi/\Omega_R$ ). In frequency maps the colour of a point represents the binding energy of its orbit with blue representing the one-third most tightly bound particles in the map, red representing the one-third least bound particles and green representing the intermediate energy range.

Since the growth of the disc makes the originally spherical distribution of particles oblate axisymmetric, the DF is entirely populated by short-axis tube orbits. In a Cartesian frequency map (Fig. 2, left-hand panel) such orbits primarily lie along a diagonal line that satisfies the condition  $\Omega_x/\Omega_z \sim \Omega_y/\Omega_z$ . Most short-axis tubes are not ‘resonant’ orbits, but in a Cartesian frequency map they all appear clustered along a line, because each short-axis tube can be viewed as arising from a radial perturbation of a parent ‘thin shell’ orbit which is a resonant orbit (de Zeeuw & Hunter 1990).

Truly resonant short-axis tube orbits are those that appear clustered along lines in a frequency map in cylindrical coordinates (Fig. 2, right-hand panel). This map shows bisymmetry between the right and left halves, reflecting the bimodal distribution in  $\Omega_\phi$  (Fig. 1). The frequency map in cylindrical coordinates shows a striking number of resonances which appear (primarily) as horizontal lines delineated by the enhanced clustering of particles at resonances between the vertical oscillation frequency  $\Omega_z$  and radial oscillation frequency  $\Omega_R$ . Resonances are seen at  $\Omega_z/\Omega_R = 0.5, 0.66, 0.75, 0.83, 1, 1.5$  (i.e.  $\Omega_z/\Omega_R = 1/2, 2/3, 3/4, 5/6, 1/1, 3/2$ ), as indicated by labels. Thus the growth of a disc in a spherical potential results in halo orbits becoming resonantly trapped at numerous resonances, primarily between the radial and vertical oscillation frequencies.



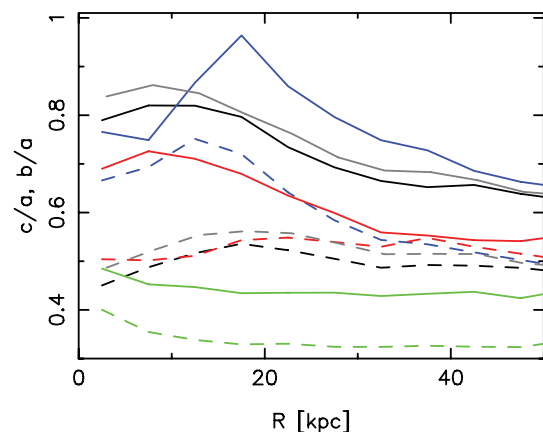
**Figure 2.** Frequency maps of  $10^4$  halo orbits in the model with a stellar disc grown in a spherical NFW halo. Left: in Cartesian coordinates the map shows that most orbits lie along the diagonal ‘resonance line’. This is not a true resonance but represents all orbits associated with the short-axis tube family. Right: frequency map of the same particles in cylindrical coordinates. The bisymmetry about  $\Omega_\phi/\Omega_R = 0$  results because the halo has no net rotation. The map shows several resonances which appear as horizontal lines: e.g.  $\Omega_z/\Omega_R = 0.5, 0.66, 0.75, 0.83, 1, 1.5$ . In both panels (and hereafter) particles are colour coded by binding energy in three energy bins, each containing one-third of the particles.

### 3.2 Triaxial haloes with rigid discs

We now consider controlled simulations of galactic potentials with triaxial haloes. The simulations are taken from D08. We consider four different orientations of the disc relative to the same triaxial halo (Halo A from D08). In all four cases the disc potential consisted of particles, but the particles were held rigid while the halo was allowed to relax. In all four cases, the inner region of the halo became less triaxial (more oblate) with short axis aligned with the symmetry axis of the disc. We define  $a, b, c$  to be the density semi-axes of the major, intermediate and short axes, respectively, of the global halo.

The shapes of the triaxial haloes were measured as described in D08. Briefly, we measure the eigenvalues of the unweighted moment of inertia tensor  $\mathbf{I}$  obtained in bins of  $N$  particles (Katz 1991, see also equation 2 from D08). The ratios of the eigenvalues of the diagonalized moment of inertia tensor ( $I_{11} > I_{22} > I_{33}$ ) are used to calculate the axial ratios  $b/a$  and  $c/a$ . The shapes were measured using an iterative procedure in annular shells of fixed semimajor axis width and are differential rather than integrated over all particles within a given ellipsoidal radius.

Fig. 3 shows the change in the axes ratios  $b/a$  and  $c/a$  as a function of radius. The green curves show that the initial triaxial halo A was very strongly prolate-triaxial (a result of low angular momentum mergers). The black curves are for model SA1 (triaxial halo + short-axis disc), the blue curves are for model LA1 (triaxial halo + long-axis disc), the red curves are for model IA1 (triaxial halo + intermediate-axis disc) and the grey curves are for model TA1 (triaxial halo + tilted disc). Regardless of the orientation of the disc, we see that its growth results in a very significant increase in axial ratios  $b/a$  (solid curves) and  $c/a$  (dashed curves) within the inner 30 kpc, for models SA1, LA1, TA1, and a moderate increase in oblateness within 50 kpc for model IA1.<sup>4</sup> (D08 shows the change in halo shape out to 200 kpc.) Model LA1 (blue curves)



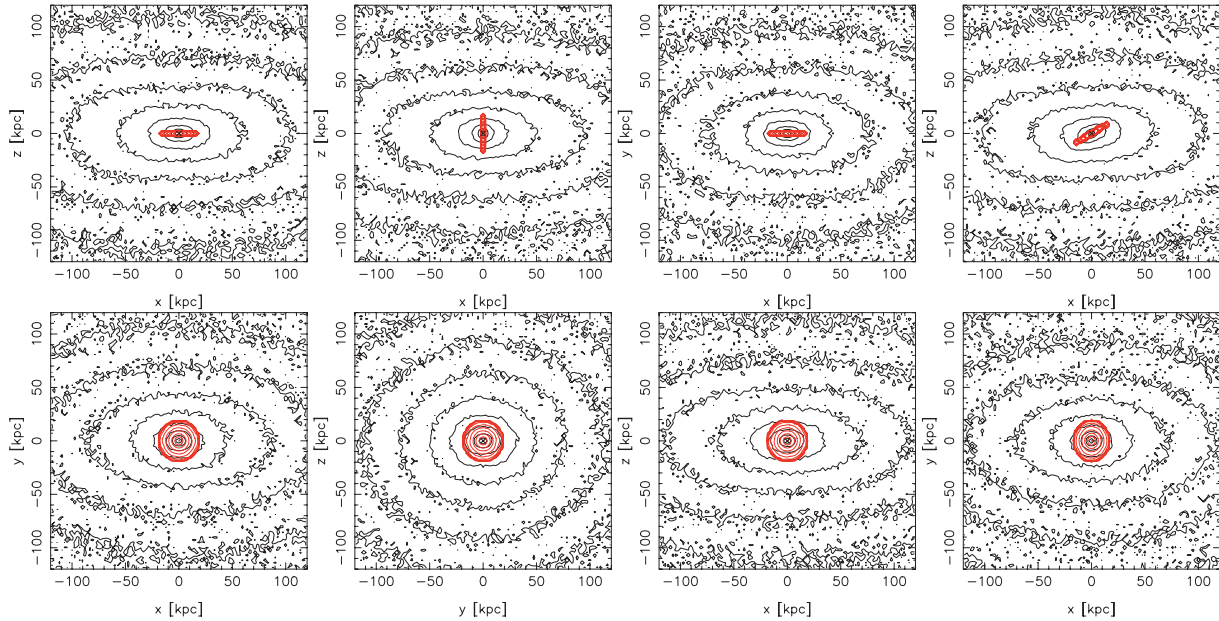
**Figure 3.** Halo axial ratios  $b/a$  (solid curves) and  $c/a$  (dashed curves) as a function of radius. The green curves show the shape of the initial triaxial halo. The black curves are for the triaxial halo + short-axis disc (model SA1), the blue curves are for the triaxial halo + long-axis disc (model LA1), the red curves are for triaxial halo + intermediate-axis disc (model IA1) and the grey curves are for the triaxial halo + tilted disc (model TA1).

shows the most significant change in shape over the radial range  $\sim 20$ –50 kpc. In the inner 15 kpc, the model is more triaxial than it is at larger radii, due to the formation of an inner elongation along the  $y$ -axis (see Fig. 4).

Although all the models become more oblate at the centre, they still remain triaxial both at small and at large radii. This is seen in Fig. 4 which shows contour plots of the dark matter halo projected density (black curves) in two projections for each of the models studied. The top row shows density contours of the dark matter particles when viewed with disc edge-on as represented by the red contours. The bottom row shows the density contours of the halo in the plane of the disc, except in model TA1 (where the disc is inclined to the principal planes). The top panels show that in all cases the contours become flattened with the symmetry axis co-aligned with

<sup>4</sup> Recall that as  $b/a \rightarrow 1$ , a model becomes more oblate.





**Figure 4.** Density contours of halo mass distribution in triaxial haloes after disc (shown by red contours) is grown with spin axis aligned in different ways. From left to right: models SA1, LA1, IA1 and TA1. Top row: density contours of haloes with disc seen edge-on. Bottom row: density contours of haloes in the plane of the disc. Halo triaxiality varies with distance from the centre of the potential.

the disc’s symmetry axis at radii  $< 15$  kpc, but retain their original elongation along the  $x$ -axis at larger radii. The lower panels show that although the halo becomes flattened in the inner region, it is not axisymmetric, even in the plane of the disc, showing that triaxiality varies with radius.

In the rest of this section we will show frequency maps of orbits in each of these models. In all cases we show frequency maps in Cartesian coordinates for  $\sim 10^4$  halo orbits selected in two ways. First, orbits were randomly selected to have an initial distance from the galactic centre  $r_g < 200$  kpc. Since these frequency maps represent a randomly selected subsample of orbits drawn from the entire dark matter halo we will refer to these maps as showing the ‘full halo DF’.

Ideally, we would like to be able to distinguish between different halo orientations relative to the disc from the DF of halo stars since the orbits of dark matter particles are not directly observable. However, halo stars are known to be significantly more centrally concentrated than the dark matter (Battaglia et al. 2005). Therefore we also present frequency maps of orbits selected in a second way, where the orbits satisfy two conditions: (i) their initial distance from the ‘Sun’  $R_s < 10$  kpc, (ii) each orbit has an apocentre radius  $r_{apo} < 50$  kpc from the galactic centre. Restriction (i) is motivated by the expectation that *Gaia* (Perryman et al. 2001) will obtain six phase-space coordinates (and in particular the most accurate distances and proper motions) for stars within 10 kpc of the Sun. Restriction (ii) is imposed because we do not anticipate that any method of halo shape determination will provide an accurate measurement of the shape of the halo at distances greater than 50 kpc from the Galactic centre. By restricting ourselves to only those stars with  $r_{apo} < 50$  kpc, we are ensuring that the maps only contain orbits which explore the region of the halo where the shape is well determined. We will refer to this second set of maps as showing the ‘inner halo DF’. Figs 3 and 4 show that the innermost region of the halo becomes significantly oblate due to the presence of the disc, regardless of orientation of the large-scale halo.

In the sections that follow we will compare the DFs of the four model by showing that the frequency map representations of their DFs differ significantly from each other regardless of whether the full halo or inner halo is represented. This is remarkable since the initial DFs of each halo (prior to the growth of the disc) were identical and Fig. 4 shows that while the disc alters the inner regions of the halo it remains triaxial and elongated along its original long axis at large radii.

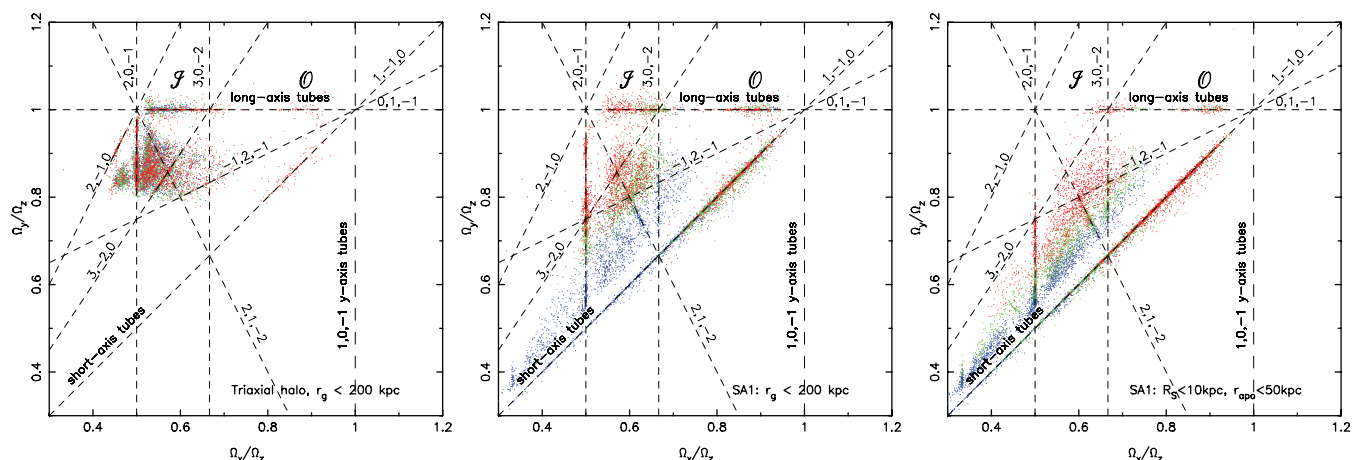
It is important to note that in all cases we begin with exactly the same set of  $10^4$  orbits from the original triaxial halo (whose frequency map is plotted in Fig. 5, left-hand panel), and follow that same set of orbits in the different models in which discs were grown adiabatically.

### 3.2.1 SA1: triaxial halo with short-axis disc

Fig. 5 (left-hand panel) shows the frequency map of a triaxial halo that was formed from multiple mergers of spherical NFW haloes. (This model is the original triaxial halo in which all the discs are grown and is referred to as ‘Halo A’ in D08 and V10.) The particles are colour coded by energy as described above.

V10 used the relationships between the fundamental frequencies of orbits in this triaxial potential to characterize them by orbital type. They showed that 86 per cent of the particles were on box orbits, 11 per cent were on long-axis tube orbits, 2 per cent were short-axis tubes and 1 per cent were chaotic. The fact that box orbits dominate over long-axis tubes and short-axis tubes in this model strongly reflects its formation history – from a two-stage binary merger, with little angular momentum. Box orbits do not have any net angular momentum and hence their three fundamental frequencies are (in general) uncorrelated with  $\Omega_x \lesssim \Omega_y \lesssim \Omega_z$ . This means that box orbits are generally found scattered in the frequency map at  $\Omega_y/\Omega_z < 1$  and to the left of the diagonal with  $\Omega_x/\Omega_y < 1$ . The long-axis ( $x$ ) tubes have  $\Omega_y \sim \Omega_z$  and hence





**Figure 5.** Cartesian frequency maps of  $\sim 10^4$  halo particles with  $r_g < 200$  kpc in the original triaxial halo (left); frequency map of the same particles after the growth of a disc perpendicular to the short-axis (model SA1) (middle). Right: Cartesian frequency map for particles with  $R_s < 10$  kpc from the ‘Sun’ and  $r_{\text{apo}} < 50$  kpc. In all panels, dashed lines mark important resonances, also labelled by resonance integers  $(l, m, n)$ ; approximate locations of inner ( $\mathcal{J}$ ) and outer ( $\mathcal{O}$ ) long-axis tubes along a horizontal line  $\Omega_y/\Omega_z \sim 1$  are marked. Short-axis tubes are along the diagonal line with  $\Omega_x/\Omega_y \sim 1$ . Particles are colour coded by energy (see text).

primarily cluster along a horizontal line at  $\Omega_y/\Omega_z = 1$ .<sup>5</sup> The inner long-axis tubes lie at smaller  $\Omega_x/\Omega_z$  values near the label ‘ $\mathcal{J}$ ’ [between the two vertical lines corresponding to the resonances  $(2, 0, -1)$  and  $(3, 0, -2)$ ] and the outer long-axis tubes lie at larger  $\Omega_y/\Omega_z$  values near the label ‘ $\mathcal{O}$ ’. The fact that this DF has only a tiny fraction of short-axis tubes (2 per cent) as determined by more rigorous orbital classification in V10 is represented by the sparse distribution of red points along the diagonal  $\Omega_x/\Omega_y = 1$ . Thin-dashed lines mark several possible resonance lines, but we see that only the two ‘banana orbit’ families labelled  $(2, -1, 0)$  and  $(2, 0, -1)$  are prominent. The qualitative dominance of the box orbits relative to the long-axis tubes and short-axis tubes can be visually assessed directly from the frequency map, without specifically going through the process of orbit classification as done by V10 and others (Carpintero & Aguilar 1998; Hoffman et al. 2009, 2010). We will see that this becomes particularly important for identifying orbits in haloes whose shapes vary with radius.

Fig. 5 (middle) shows the frequency map of the same set of halo orbits after the growth of a stellar disc perpendicular to the short axis of the halo (model SA1). V10 found that the fraction of box orbits has now decreased from 86 per cent in the original triaxial halo to 48 per cent after the growth of the disc, but the long-axis tube fraction remained almost the same (12 per cent). They found a significant increase in the fraction of short-axis tubes (from 2 to 31 per cent) and chaotic orbits (from 1 to 9 per cent). The increased fraction of short-axis tubes relative to the long-axis tubes and box orbits is obvious in the frequency map, where the short-axis tubes appear as the enhanced clustering along the diagonal line. The map shows that the disc also causes orbits to become separated into distinct bands in energy (colour), with the most tightly bound (blue) particles migrating to the bottom-left corner.

The migration of particles in the frequency map occurs because the introduction of the disc potential perpendicular to the  $z$ -axis increases the vertical frequency ( $\Omega_z$ ) of orbits more significantly than either of the other two frequencies. Since the disc is centrally

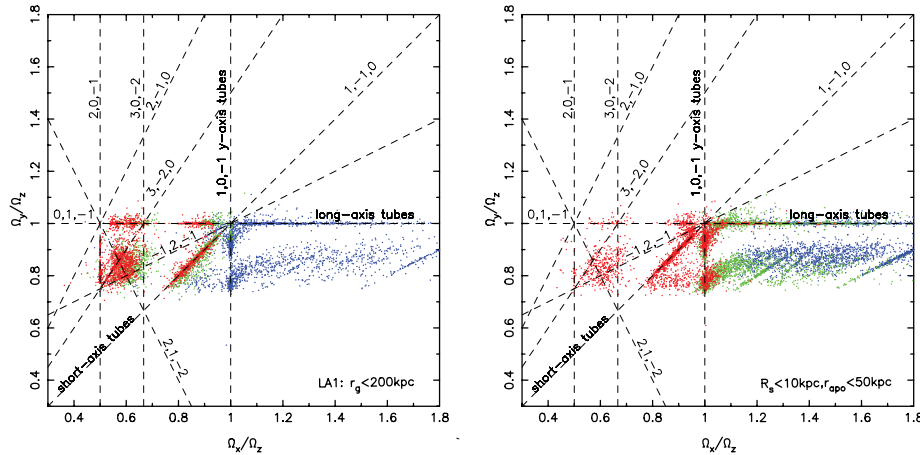
concentrated, the increase in  $\Omega_z$  is particularly significant for orbits which are deeper in the potential (coloured blue). Therefore both  $\Omega_y/\Omega_z$  and  $\Omega_x/\Omega_z$  decrease, and the blue points move downwards and to the left. The growth of a disc also increases the fraction of halo orbits trapped in resonances, but some resonances are destroyed. For instance, the vertical  $(2, 0, -1)$  ‘banana resonance’ has significantly increased in prominence (due to trapping of orbits in the  $xz$  plane), but the  $(2, -1, 0)$  banana resonance in the triaxial model (which lies in the  $xy$  plane) is destroyed because this is the disc plane, and the presence of the disc decreases the degree of triaxiality. New resonances are also populated e.g.  $(3, 0, -2)$  the ‘fish’ resonance, and  $(2, 1, -2)$  resonance.<sup>6</sup> Since a frequency map represents the *ratios* of the frequencies and not the frequencies themselves, it is insensitive to the absolute value of the energy of individual particles and it is therefore possible to identify the *global* orbital families and resonances (i.e. those that are important over a large range of energies).

Fig. 5 (right-hand panel) shows a frequency map of  $\sim 10^4$  inner halo particles. A comparison of the right-hand and middle panels of Fig. 5 shows that the main features of the frequency map of the entire halo (middle) are also seen in the map of inner halo orbits (right). Thus, although the halo is more oblate within 50 kpc than at larger radii, the orbits in the inner halo share the major orbit families and resonances of the entire halo.

The reader may be concerned that since the halo is moderately triaxial and not axisymmetric, by selecting orbits within a ‘torus’ of radius  $R_s$  rather than in a region localized around the ‘Sun’, the frequency maps of stars in the entire torus would not reflect variations in the phase-space structure at different locations in the equatorial plane. We therefore also produced frequency maps of subsets of the full sample of  $10^4$  orbits, which contained only those orbits with initial positions within an individual quadrant of the galaxy model, with quadrants symmetrically about either the major or minor axes. These frequency maps are not shown since they are virtually indistinguishable from those of the full samples of  $10^4$  orbits (shown in the middle and right-hand panels) demonstrating

<sup>5</sup> Long- and short-axis tubes generally are not considered ‘resonant’ orbits; however they may be viewed as perturbations of their respective ‘thin shell’ orbits (de Zeeuw & Hunter 1990).

<sup>6</sup> Three-dimensional images of all the major resonances in triaxial potentials are shown in Merritt & Valluri (1999).



**Figure 6.** Left: frequency map of  $10^4$  halo orbits in LA1 with  $r_g < 200$  kpc (the frequency map of these orbits in the original triaxial halo is shown in Fig. 5). Axes  $x, y, z$  chosen to be the *global* long, intermediate and short axes of the halo. Right: frequency map of  $\sim 10^4$  halo orbits with  $R_s < 10$  kpc and apocentres less than 50 kpc.

that within the ‘solar’ circle, the halo is sufficiently oblate that the orbital populations do not depend significantly on azimuthal location.

### 3.2.2 LA1: triaxial halo with long-axis disc

It has been conjectured that instead of the MW halo being oblate with short-axis co-aligned with the spin axis of the disc (as in model SA1), the disc may in fact be prolate and perpendicular to the long axis of the halo. Helmi (2004) concluded, from modelling the tidal disruption of Sagittarius dwarf satellite, that the kinematics of stars forming the leading arm of the Sgr stream suggest that the dark matter halo may be prolate with an average density axial ratio close to  $5/3$  with long axis perpendicular to the disc. This orientation has also been suggested by studies of distribution of dark matter subhaloes that are satellites of MW-sized dark matter haloes analysed in cosmological  $N$ -body simulations (Zentner et al. 2005).

We investigated this possibility in simulation LA1, where a disc was grown perpendicular to the long axis of the halo. The projected density contours of halo particles in simulation LA1, after the halo relaxed into a new equilibrium with the disc potential, are shown in Fig. 4 (second column from left). The density contours show that the halo triaxiality varies with radius. At small radii ( $x < 20$  kpc) the long axis is in the disc plane, but at larger radii the long axis remains perpendicular to the disc.

Fig. 6 (left-hand panel) shows a frequency map of  $10^4$  halo orbits randomly selected to lie with  $r_g < 200$  kpc (representing the entire halo DF). Recall that the frequency map of original triaxial halo for this model is shown in Fig. 5 (left-hand panel). In this map, a significant fraction of orbits deep inside the potential (blue) populate the horizontal (1:1) resonance line corresponding to the global long axis of the halo. Long-axis tube orbits circulate about the  $x$ -axis in a fixed direction. The dramatic increase in the length and strength of this family of tube orbits as well as their location on the frequency map (relative to the fraction long-axis tube orbits in the original triaxial halo in Fig. 5, left-hand side) is a direct consequence of the growth of the disc potential. In this simulation the disc is symmetric about the long axis. Therefore orbits (especially those deeper in the potential) experience a somewhat larger increase  $\Omega_x$ , than in the

other two frequencies. The greater increase in  $\Omega_x$  causes the blue points on the frequency map to migrate towards the right of the map. Not all the migration to the right is associated with long-axis tubes; we see a significant increase in the fraction of box orbits (below the horizontal line) as well as several distinct resonances in this population.

We also note that many tightly bound (blue) points clustered along the vertical line corresponding to the intermediate ( $y$ ) axis tubes. The ‘intermediate-axis’ tube family is generally expected to be unstable. However the strong clustering along the vertical line shows that this family is both stable and well populated in this potential. The reason for this can be seen in Fig. 4 (bottom row in second column from left), which shows a slight elongation of the dark matter (black contours) density along the  $y$ -axis for  $x < 10$  kpc. It appears that in this model the  $y$ -axis tubes appear for the most bound population because this is a *local long axis*!

In model LA1 there is a large increase in the fraction of tightly bound (blue) orbits associated with the  $x$ -axis tube family, which happens to coincide with the symmetry axis of the disc. This is in sharp contrast with Fig. 5 where the disc, oriented along the  $z$ -axis, caused an increase in the fraction of tightly bound orbits associated with the ( $z$ ) short-axis tube family. There are also short-axis tube orbits in LA1 but most are weakly bound (red). Thus although the initial triaxial haloes were identical in shape and DF, the differences in the orientation of their disc relative to the halo resulted in very different orbit populations (i.e. DFs) especially for the more tightly bound particles.

In Fig. 6 (right-hand panel) we show orbits in the inner halo ( $R_s < 10$  kpc,  $r_{apo} < 50$  kpc). The frequency map of the inner halo (right) is very similar to that of the outer halo (left), with the main difference being a decrease in the fraction of box orbits between the resonance lines (2,0,−1) and (3,0,−2). The colours corresponding to the energies of particles change purely because the range of energies in the right-hand plot is smaller. The box-orbit resonances (below the long-axis tubes and to the right of the  $y$ -axis tubes) are seen more clearly, because this region is more densely populated with orbits due to the orbit selection criterion.

In both SA1 and LA1, the inner haloes are significantly flattened in the plane of the disc (although they remain triaxial) at  $r < 50$  kpc. It is therefore remarkable that the frequency maps of orbits confined

to the inner haloes of these two potentials (right-hand panels of Figs 5 and 6) reflect the differences in large-scale orientation of the haloes relative to the orientations of their discs. Although there are clearly differences between the maps of the inner haloes and the full haloes, these differences are significantly less than the differences that arise from the different orientations of the disc. Recall that the initial haloes were identical prior to the growth of the discs. *This implies that when accurate phase-space coordinates for stars are obtained from future survey such as Gaia, they can be used to infer the global shape of the halo relative to the disc, even if accurate coordinates are only obtained within 10 kpc of the Sun.*

### 3.2.3 TA1: triaxial halo with tilted disc

We now consider a model in which the disc was grown inclined to the  $xy$  plane of the triaxial halo (model TA1). Such an orientation is motivated by studies of dark matter haloes from cosmological  $N$ -body simulations that show that the relative orientation of the angular momentum axis of triaxial haloes is on average no more than  $25^\circ$ – $30^\circ$  from the short axis of the triaxial halo (Bailin & Steinmetz 2005). Furthermore, numerous simulations show that the disc can be misaligned with the symmetry axis of the halo (van den Bosch et al. 2002).

Fig. 7 (left-hand panel) shows the frequency map of  $10^4$  halo particles with  $r_g < 200$  kpc. This frequency map is very similar to the model with the short-axis disc (Fig. 5, middle panel), but also shows a clustering of (blue) points along a vertical line at  $\Omega_x/\Omega_z = 1$  (corresponding to  $y$ -axis tubes). This family of orbits arises because the disc axis in this model is inclined to the  $xy$  plane such that the  $x$ -axis is in the plane of the disc but the  $y$ -axis is at an angle of  $30^\circ$  to the disc. The disc potential therefore induces resonant trapping of tightly bound halo orbits causing a larger circulation in the plane of the disc, and consequently more angular momentum about the  $y$ -axis. Although this family is rotating about an axis inclined to the  $y$ -axis, this trapping appears on the Cartesian frequency map as an enhanced clustering of tightly bound orbits about this axis. This family of intermediate-axis tubes does not appear in the most tightly bound orbits of any other model. Unlike in model LA1, the density contour plots do not show a noticeable elongation along the  $y$ -axis.

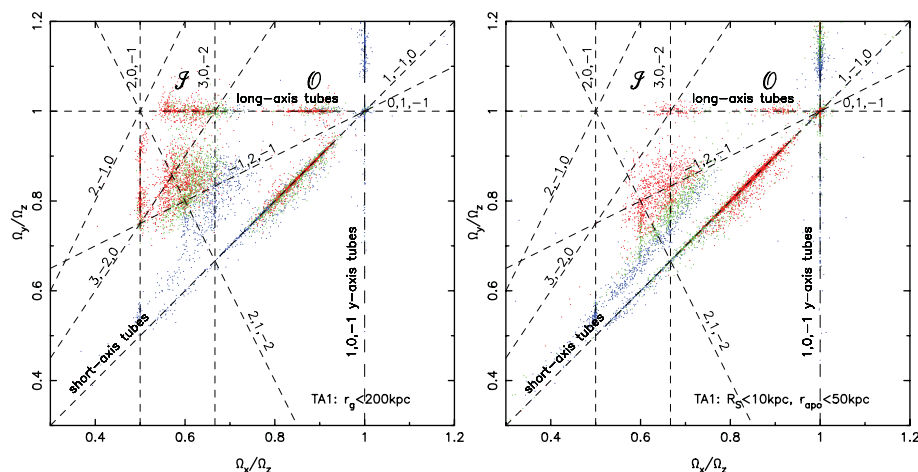
The DF of the inner halo of TA1 (Fig. 7, right-hand panel) displays many of the features of the full DF within 200 kpc (middle), with some differences: the inner long-axis tube family is less prominent, the banana family  $(2, 0, -1)$  is very sparsely populated in the inner region and a new resonance family is seen at  $\Omega_x/\Omega_z = 0.6$ . Apart from these differences, the important major orbit families (the long-axis tubes, short-axis tubes and intermediate-axis tubes) are well represented, showing that the DF of the inner halo, while different from that of the entire halo, shares the most important characteristics that distinguish it from other models.

Fig. 3 shows that the shape of the inner halo of TA1 (grey curves) is very similar within 50 kpc to the shape of the inner halo of SA1 (black curves). However, the frequency map of the inner halo of TA1 (Fig. 7, right-hand panel) is quite different from that of SA1 (Fig. 5, right-hand panel). This implies that their DFs are very different – again a consequence of the different original orientation of the discs relative to their globally triaxial haloes. Thus, our analysis of the orbital phase-space structure using the frequency maps allows us to gain insights into the differences in the orbital properties and DFs of two haloes with very similar inner halo shapes.

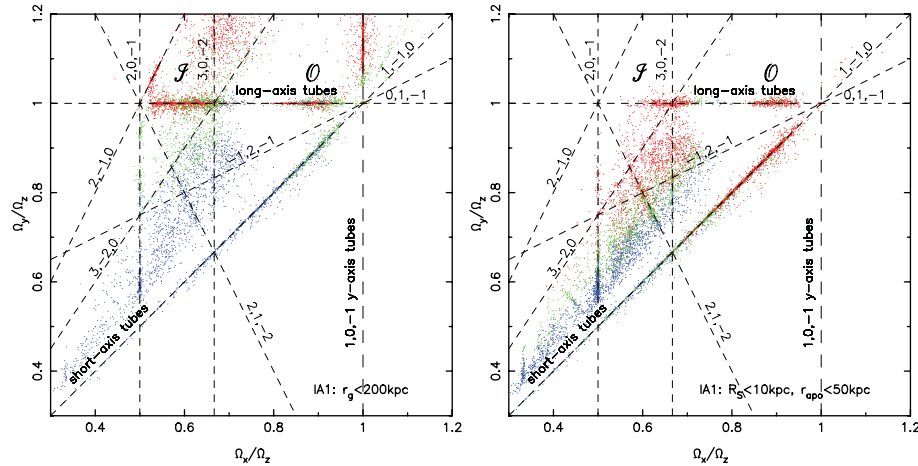
### 3.2.4 IA1: triaxial halo with intermediate-axis disc

Recently, Law and collaborators (Law, Majewski & Johnston 2009; Law & Majewski 2010) remeasured the shape of the MW halo by fitting both the velocities and positions of stars in the Sagittarius tidal stream with a triaxial halo model. They find that a slightly triaxial halo with the Sun located roughly along the minor axis gives the best fit to the available kinematic and positional data of stream stars. The best-fitting configuration requires the MW disc to be perpendicular to the intermediate axis of the triaxial halo – a disc configuration that is believed to be inherently unstable (Heiligman & Schwarzschild 1979).

Our final model simulates this configuration with a rigid disc potential grown perpendicular to the intermediate axis of the triaxial halo [models investigating the stability of haloes with live discs will be presented in Debattista et al. (in preparation)]. The frequency map of  $10^4$  randomly selected halo orbits with  $r_g < 200$  kpc is shown in Fig. 8 (left-hand panel). The map shows a prominent clustering of weakly bound orbits along the vertical line corresponding to intermediate-axis tubes with  $\Omega_x/\Omega_z = 1$  at values of  $\Omega_y/\Omega_z > 1$ .



**Figure 7.** Left: frequency map of  $10^4$  halo orbits with  $r_g < 200$  kpc after the disc grows tilted at an angle of  $30^\circ$  to the  $xy$  plane of the triaxial potential (model TA1) (the frequency map of these orbits in the original triaxial halo is shown in Fig. 5, left-hand panel). Right: about  $10^4$  orbits selected from the ‘solar’ neighbourhood and confined to the inner halo of the model.



**Figure 8.** Left: frequency map of  $10^4$  halo orbits with  $r_g < 200$  kpc after the disc grows perpendicular to the intermediate ( $y$ ) axis in the triaxial potential (model IA1). The frequency map of these orbits in the original triaxial halo is shown in Fig. 5 (left). Right: about  $10^4$  orbits from the inner halo of the same model. The  $y$ -axis tube family is populated by weakly bound (red) points and is not important in the solar neighbourhood map on the right.

Since the disc is symmetric about the  $y$ -axis, at small radii the halo does become more oblate, with  $y$  as its symmetry axis. However we see that even the weakly bound particles (red), which are at large radii and expected to be less affected by the disc, are also associated with the intermediate ( $y$ ) axis tube family. Fig. 3 shows that although the shape of halo SA1 (black curves) is more oblate than halo IA1 (red curves), the radial profiles (of  $b/a$  and  $c/a$ ) are very similar for these two haloes (except in the absolute degree of flattening). Consequently, when we plot only those orbits which are confined to the inner halo (Fig. 8, right-hand panel), we see that the frequency maps of IA1 and SA1 (Fig. 5, right-hand panel) are so similar that they are hard to distinguish from each other. The intermediate-axis tube family that was seen in Fig. 8 (left-hand panel) has completely disappeared, showing that all the orbits that made up this family were part of the outer halo. However, this similarity in the DFs is not entirely surprising since Fig. 4 shows that SA1 and IA1 have similar density contour distributions (this similarity was also found by D08).

### 3.2.5 Discussion of disc-halo orientation effects

Figs 3 and 4 showed that the growth of a disc galaxy in a triaxial dark matter halo of arbitrary orientation modifies the shape of the inner part of halo, but leaves the outer part largely unaffected. However, most methods for determining the shape of the halo assume that dark matter is stratified on concentric similar ellipsoids of constant shape (i.e. not varying with radius). With this assumption the shape of the halo can be measured with an accuracy of a few per cent, out to 50–70 kpc (Johnston, Hernquist & Bolte 1996; Gnedin et al. 2005). Since the shape of the halo probably varies significantly with radius due to disc formation, the assumption of a constant halo shape is not valid.

Although the radial variation of the shape of the halo will be impossible to measure, the fact that in all cases the inner halo is nearly oblate and flattened like the disc will enable us to use halo orbits to constrain both the shape and the DF of the inner halo.

Furthermore, the analysis of the DFs of four different halo models showed that a frequency map provides detailed information on the various orbit families that constitute the halo DF and relative abundance of the different orbit families at various energies. In ad-

dition, while the disc potential traps tube orbits, which share the disc symmetry axis, even restricted maps of halo orbits confined to the inner region ( $r_{\text{apo}} \sim 50$  kpc) show all the orbit families present in the global halo.

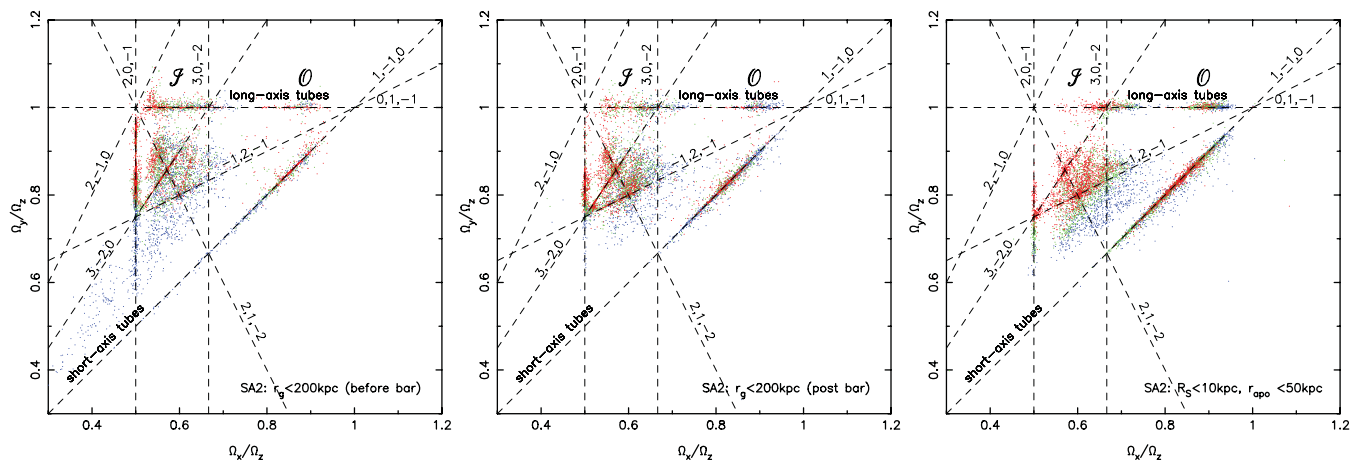
It is worth noting that using the frequency mapping method for identifying orbit families is superior to methods that rely on orbital properties (Carpintero & Aguilar 1998; V10; Deibel et al. 2011) because the latter methods assume a priori that the shape of the potential is constant with radius. In triaxial potentials with constant shape, intermediate-axis tubes are unstable and not expected in the DF (Heiligman & Schwarzschild 1979), yet we see from the frequency maps that three of four models contained members of this family, because the triaxiality of the halo varies with radius. To build a DF for the stellar halo, it is important to have a full representation of all the orbit types that are important in the halo, since the orbit populations, in turn, reflect the large-scale shape, orientation and formation history of the Galactic halo.

We caution that in all the simulations given above, the disc particles were held fixed and did not dynamically respond to the change in the halo. Theoretical arguments indicate that at least some of these orientations, e.g. IA1 (Heiligman & Schwarzschild 1979), are likely to be dynamically unstable. A more detailed analysis of self-consistent dynamical models of such systems is needed to ascertain whether such disc orientations would be found in nature (Debatista et al., in preparation).

### 3.3 Non-axisymmetric haloes with live discs

Most stellar discs, including that of the MW, are not axisymmetric or time-independent since they contain features such as spirals and bars. These non-axisymmetric features drive angular momentum exchanges. The angular momentum exchange that occurs between a rotating bar and the dark matter (and presumably also the stellar halo) is expected to result in a change in the angular momentum of halo particles that interact with the bar (Weinberg 1985; Debatista & Sellwood 2000). In this section we study the effects of angular momentum and energy transfer resulting from coherent time-dependent perturbations from a bar. We also studied two models in which a stellar and gaseous disc forms via realistic processes in a spherical halo and in a prolate halo. In these simulations,





**Figure 9.** Left: frequency map of  $10^4$  halo orbits with  $r_g < 200$  kpc after a short-axis disc is grown in the triaxial halo (model SA2) before the formation of a bar. The frequency map of these orbits in the original triaxial halo is shown in Fig. 5 (left-hand panel). The disc of live particles rapidly forms a bar, which subsequently dissolves. Middle: frequency map of the same halo orbits after the bar has dissolved. Right: frequency map of  $10^4$  orbits in the inner halo after the bar has dissolved.

hot gas in the initial halo which has angular momentum cools to form a rotating disc followed by star formation and feedback (see Section 2 for details).

### 3.3.1 Triaxial halo with barred disc

To study the effects of a bar, we adiabatically grew a stellar disc that had only 30 per cent of the mass of the disc in the previous simulations inside a triaxial halo (with symmetry axis parallel to the short axis of the halo, model SA2). Because of its low mass, the bar that forms in this system is eventually destroyed by its interaction with the triaxial halo (Berentzen & Shlosman 2006). Fig. 9 (left-hand panel) shows the frequency map of  $10^4$  halo particles with  $r_g < 200$  kpc after the disc had reached full mass. The frequency map shows the two long-axis tube families (marked with script ‘ $\mathcal{J}$ ’ and ‘ $\mathcal{O}$ ’), a strong short-axis tube family and several box-orbit resonances [e.g.  $(2, 0, -1)$ ,  $(2, 1, -2)$ ,  $(3, -2, 0)$ , as well as a few other unlabeled resonances seen mostly in red points].

After the disc reached its full mass the particles were made ‘live’, i.e. the disc was allowed to evolve self-consistently along with the halo. The rotating disc rapidly formed a bar and spiral patterns. The bar survived for  $\sim 10$  Gyr and finally dissolved. Note that we always integrate orbits in a potential without a bar. This is because we ‘freeze’ the potential before integrating orbits, and the presence of a bar would result in an unrealistic ‘freezing’ of a time-dependent rotating bar. Most models that seek to obtain the DF of the MW disc and halo (e.g. Binney 2010; Binney & McMillan 2011) neglect the bar since it is difficult to model its time-dependent potential.

Fig. 9 (middle panel) shows a frequency map of the same set of  $10^4$  halo particles with  $r_g < 200$  kpc, while the right-hand panel shows  $10^4$  halo particles in the inner halo, after the growth and dissolution of the bar. This frequency map shows that tightly bound (blue) particles that were associated with the  $(2, 0, -1)$  resonance and the short-axis tube family are scattered to other parts of the map. This is because these orbits are most strongly affected by the central bar. The bar exchanges angular momentum with resonant halo particles (Tremaine & Weinberg 1984; Debattista & Sellwood 2000; Athanassoula 2002). Thus the blue (tightly bound) halo particles in the bottom right of the left-hand panel are scattered and the resonances at the bottom left of the map are no longer seen.

Many of the blue points now appear to be associated with the outer long-axis tube family (at  $\Omega_x/\Omega_z \sim 0.9$ ) and the short-axis tube family. Resonances associated with less bound (red) particles have become broader or disappeared. Other resonance lines populated by the weakly bound (red) particles are only slightly affected (e.g.  $2, 0, -1$  and  $3, -2, 0$ ). These tightly bound particles are seen more clearly in the right-hand panel which shows orbits from the inner halo.

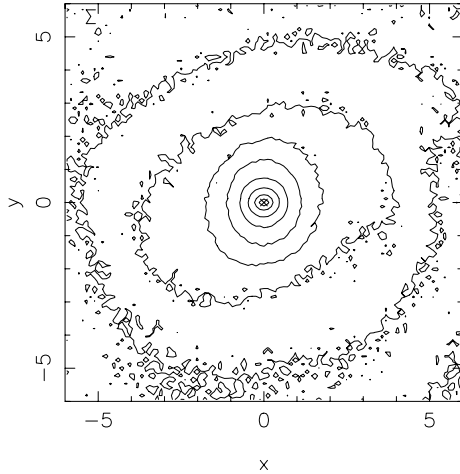
It is clear that although the middle and right plots have fewer strong resonances than the halo prior to the bar (left), the major orbit families still appear with the same relative strengths. Although some resonances appear broader and those associated with particles deeper in the potential have been scattered to other locations on the map, the overall structure of the frequency maps is similar to SA1 which is also a model with a disc rotating about the short axis of triaxial halo. This shows that although time-dependent perturbations from a bar can affect the shape and DF of the halo, many resonances survive, especially those that are populated by orbits that do not interact strongly with the bar, and the overall structure of the frequency map is characterized by the global nature of the potential.

### 3.3.2 Haloes with live discs of gas and stars

We now study frequency maps of halo orbits from models where a stellar disc forms from hot gas with some initial angular momentum (defined to be about the  $z$ -axis) in a spherical halo (model SNFWgs) and in a prolate halo (model SBgs) (see Section 2.1 for details). Following the formation of the disc, the dark matter halo became fairly oblate in model SNFWgs, while the disc develops only a mild oval distortion (see Fig. 10). In these simulations no stars form in the halo and therefore we once again study the orbits of dark matter particles.

Fig. 11 (left-hand panels) shows the frequency maps in Cartesian coordinates of  $10^4$  halo stars selected with  $r_s < 200$  kpc in the initial spherical halo (top) and in the initial prolate halo (bottom). As we saw in several previous models (e.g. SA1, TA1), the growth of the disc results in an increase in the fraction of short-axis tubes deep in the potential (blue points along the diagonal). Both models show





**Figure 10.** Density contours of disc that forms in the spherical halo from cooling halo gas. The disc develops a mild oval distortion ( $x$ - and  $y$ -axes coordinates are in kpc).

prominent short-axis tube families which manifest as the strong clustering along the diagonal.

V10 showed that prolate haloes are dominated by long-axis tubes which persist even after the growth of a baryonic component, but these long-axis tube orbits become ‘rounder’. Long-axis tubes are seen in the prolate halo as the clustering of points along the horizontal line with  $\Omega_y/\Omega_z = 1$  (bottom-left panel). Surprisingly the spherical halo also shows long-axis tubes (predominantly in the tightly bound orbits coloured blue). This is likely to be because the disc forms a slight oval distortion (see Fig. 10). Compared to the originally triaxial models SA1, LA1, IA1 and TA1, these two models have only a small fraction of box orbits (scattered points in the middle of the map) because these models are not triaxial.

It is also instructive to analyse the same set of halo orbits in cylindrical coordinates (with  $z$  as the symmetry axis).<sup>7</sup> The two right panels of Fig. 11 show frequency maps in cylindrical coordinates. Several resonances previously seen in Fig. 2, such as  $\Omega_z/\Omega_R = 0.5, 1$ , are seen in both the spherical halo and the prolate halo. In addition both show new resonances  $\Omega_\phi/\Omega_z = \pm 0.5, 1$ . The prolate halo (bottom right) also shows a large fraction of particles associated with the vertical resonance line at  $\Omega_\phi/\Omega_R = \pm 0.5$ , which are long-axis tubes.

The disc formation process in this subsection is dynamically quite different from that in the previous section where rigid discs made up of particles were adiabatically grown in place. These two simulations confirm that resonant trapping of halo orbits occurs even during the more realistic dissipative processes by which real stellar discs form.

#### 4 USING FREQUENCY ANALYSIS TO CONSTRAIN POTENTIAL PARAMETERS

In the simulations so far, we have integrated orbits in the total  $N$ -body potential for the galaxy model, which was known perfectly. Since the orbits were integrated in the self-consistent  $N$ -body potential from which they were drawn, the frequency maps obtained represent the true DFs of these galaxy models. Since a major goal

of current and future galactic surveys is to obtain both the potential and DF of the Galaxy (Binney 2010; Binney & McMillan 2011), our frequency-based method is a promising approach for doing this. However, in reality the potential of the Galaxy is not known and it will also be measured from the spatial and kinematic distribution of stars. Ideally both the self-consistent DF and the potential will be recovered from six phase-space coordinates for large numbers of stars (Binney 2010; Binney & McMillan 2011).

The most promising methods for measuring the potential of the halo use kinematics of stars in tidal streams (Johnston et al. 1999; McMillan & Binney 2008), or accurate orbits of hyper-velocity stars (Gnedin et al. 2005). While these methods are expected to yield excellent estimates of the shape and density profile of the MW halo if it is stratified on concentric ellipsoids, all current numerical experiments show that the halo’s shape varies with radius, making the measurement of the shape of the halo at all radii challenging due to the absence of coherent tidal streams and/or hypervelocity stars over a range of radii.

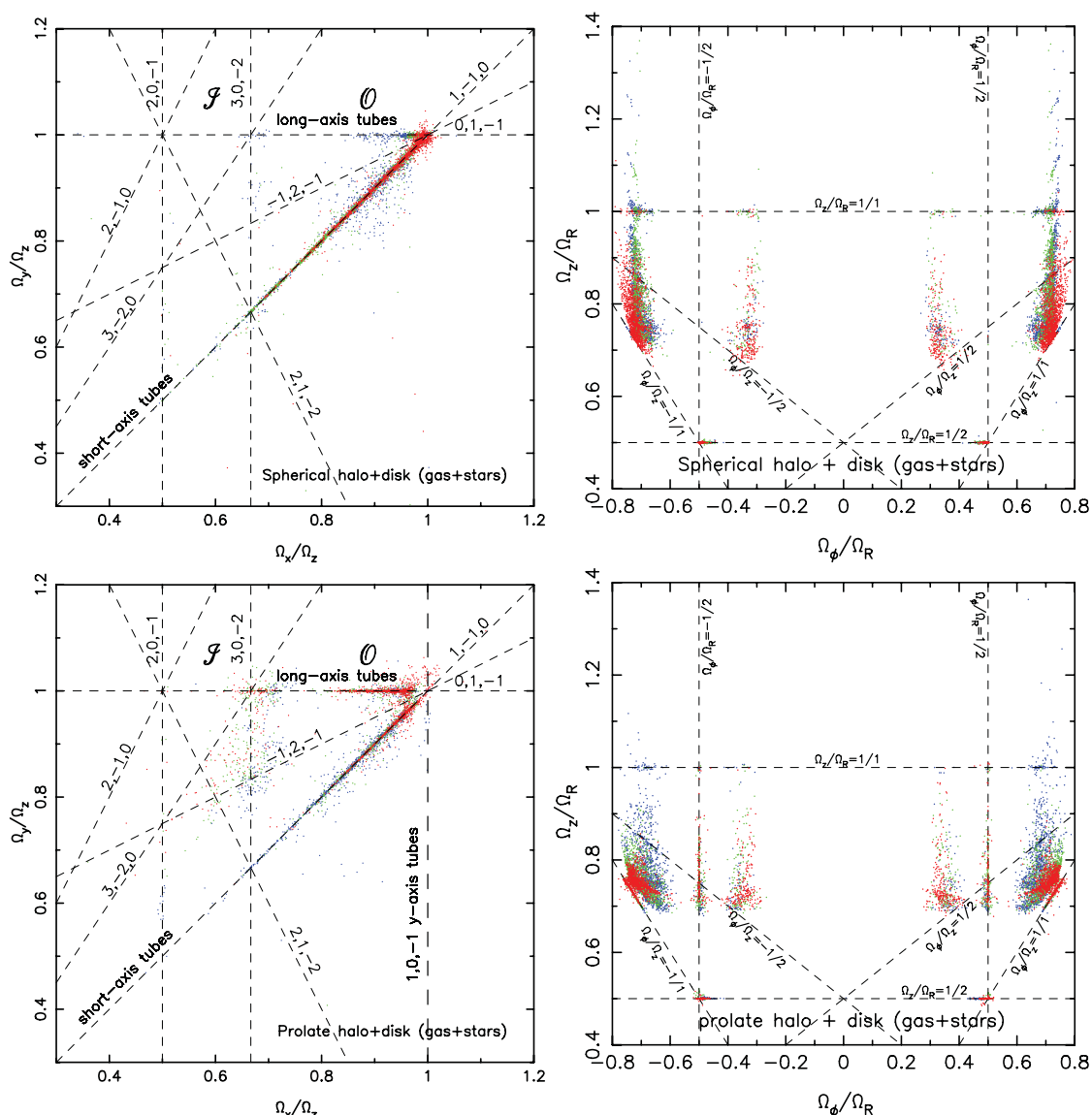
Halo stars are, however, distributed over a wide range of radii and current studies show that even local halo stars have enough kinetic energy to travel to large radii (Carollo et al. 2010). We now test whether it will be possible to gain information about the true potential and DF from the frequency analysis of a large number of such halo stars.

Jeans Theorem states that any steady-state equilibrium DF depends on phase-space coordinates only though the integrals of motion (Binney & Tremaine 2008). This implies that for a DF (or a random subsample thereof) that is in self-consistent equilibrium with its background potential, only a small fraction of orbits are irregular or chaotic. If a large fraction of orbits are chaotic (i.e. not confined to regular tori in phase space), they are expected to diffuse in phase space causing the DF to evolve until the system reaches a new equilibrium (Merritt & Valluri 1996). If the degree of chaoticity is low (either there are only a small number of strongly chaotic orbits or there are many weakly chaotic orbits) the evolution of the potential could take some time and fairly long-lived quasi-equilibria can exist (Poon & Merritt 2004). However, if the initial conditions of a large number of orbits are strongly out of equilibrium with the background potential, the overall degree of chaoticity (or ‘chaotic momentum’) is large and the self-consistent system will evolve rapidly (Kalapotharakos 2008).

In our experiments each halo particle is treated as a test particle which is integrated in a frozen background potential, and hence the orbits do not self-consistently influence the frozen potential. None the less, the fact that we use a large ( $1\text{--}2 \times 10^4$ ) ensemble that is a random sampling of the halo DF gives us additional collective power. A DF that is not in self-consistent equilibrium with the background potential will relax. This relaxation manifests as orbital diffusion or mixing (Valluri et al. 2007). This mixing will occur whether the potential is time independent or varying, so long as the DF is not in self-consistent equilibrium with the potential. We hypothesize that the diffusion rates  $\log(\Delta f)$  can be used to measure the rate of diffusion of mixing of orbits in ensemble that is evolved in a selected potential. If the DF is out of equilibrium we should obtain larger diffusion rates. We test this hypothesis by comparing the distributions of the diffusion rates of ensembles of orbits evolved in different potentials.

Our objective is to distinguish quantitatively (with an assigned statistical confidence) between the correct potential and incorrect potentials using halo stars for which all six phase-space coordinates are available. We computed diffusion rates for all  $10^4$  orbits selected from three models SA1, LA1 and IA1, when the orbits

<sup>7</sup> This coordinate system is not the proper one in which to examine long-axis tubes, which require cylindrical coordinates where  $x$  is the axis of symmetry.



**Figure 11.** Frequency maps of  $10^4$  halo particles with  $r_g < 200$  kpc in two haloes in which hot halo gas cools into a star-forming disc. Left: frequency maps of halo orbits in Cartesian coordinates; Right: frequency maps of the same particles in cylindrical coordinates. Top row: maps of halo orbits in the case when the initial halo was spherical NFW; the bottom row shows frequency maps in the case when the initial halo was prolate axisymmetric. Both Cartesian maps show short-axis tubes and long-axis tubes, and only a few box orbits. Cylindrical maps show several resonances as indicated by thin-dashed lines and labels.

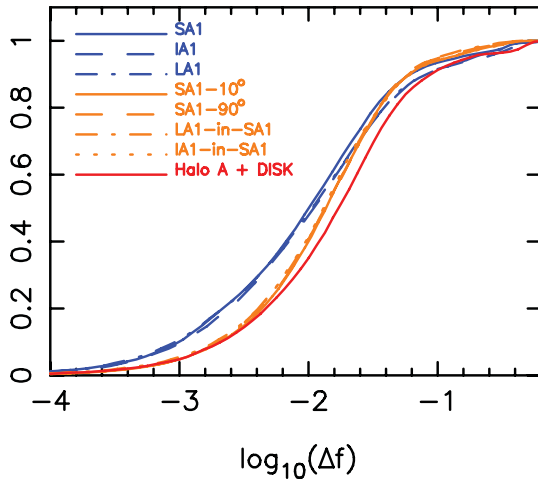
were integrated in the correct potential. We also integrated the orbits from the DF of SA1 in two ‘slightly incorrect’ potentials: SA1 rotated about the  $z$ -axis by  $10^\circ$  (model SA1- $10^\circ$ ) and by  $90^\circ$  (model SA1- $90^\circ$ ). Two other ‘slightly incorrect’ potentials consisted of  $10^4$  orbits drawn from the DF of models LA1 and IA1 and integrated in the potential for SA1 (models LA1-in-SA1 and IA1-in-SA1, respectively).

We also considered one ‘strongly incorrect’ model: the orbits of the initial triaxial halo A were integrated in a potential consisting of halo A + a short-axis disc. Since the halo was not allowed to relax in response to the presence of the disc, the orbits are strongly out of equilibrium since, in effect, the disc potential is ‘turned on suddenly’.

Fig. 12 shows the cumulative distribution functions (CDFs) of the diffusion parameters for ensembles  $10^4$  orbits from the three DFs from SA1, LA1 and IA1 evolved in their own potentials as blue curves. The CDFs of diffusion rates for  $10^4$  orbits in each of the

four ‘slightly incorrect’ potentials, SA1- $10^\circ$ , SA1- $90^\circ$ , LA1-in-SA1 and IA1-in-SA1 (orange curves), and the ‘significantly incorrect’ potential (red curve). The clear separation of the curves shows that the CDFs of  $\log(\Delta f)$  for ensembles integrated in the correct potential (blue curves) are always to the left of the orange curves implying that in the correct potentials there are significantly more orbits with low diffusion rates [ $\log(\Delta f) \lesssim -2$ ] than when these ensembles are evolved in incorrect potentials (orange and red curves).

We carried out pair-wise comparisons of the eight CDFs in Fig. 12 using the non-parametric Kolmogorov–Smirnov (KS) test to evaluate the probability  $p$  that the distributions are statistically different from each other. We compute the KS statistic which measures the ‘distance’  $d$  between the two distributions. Small distances  $d$  between pairs of distributions and low (or zero) values of  $p$  indicate a low probability that the two distributions are (statistically) identical. Results of representative subsets of the tests are reported in Table 2. The first three tests in the table compare the CDF of orbits from



**Figure 12.** CDFs of orbital diffusion rates  $\log(\Delta f)$  for ensembles of  $10^4$  orbits which were evolved in the correct potential (blue curves), evolved in slightly incorrect potentials (orange curves) and in a strongly incorrect potential (red).

**Table 2.** Results of KS tests comparing CDFs of  $\log(\Delta f)$  from Fig. 12 pairwise.

Models compared in KS test	Distance $d$	Probability $p$
SA1 versus SA1-10	0.11	0.0
SA1 versus LA1-in-SA1	0.10	$4.2 \times 10^{-44}$
SA1 versus Halo A + DISC	0.15	0.0
LA1 versus LA1-in-SA1	$9.9 \times 10^{-2}$	$8.6 \times 10^{-43}$
IA1 versus IA1-in-SA1	$8.6 \times 10^{-2}$	$2.7 \times 10^{-26}$
SA1-90 versus SA1-10	$1.2 \times 10^{-2}$	0.45
SA1-10 versus LA1-in-SA1	$1.7 \times 10^{-2}$	0.12
LA1-in-SA1 versus IA1-in-SA1	$1.9 \times 10^{-2}$	0.12
SA1-90 versus Halo A + DISC	$8.8 \times 10^{-2}$	$1.7 \times 10^{-34}$
LA1-in-SA1 versus Halo A + DISC	$8.1 \times 10^{-2}$	$1.9 \times 10^{-29}$

the SA1 DF evolved in the correct potential with the same orbits evolved in three incorrect potentials. The small values of  $d$  and  $p$  indicate a very low probability that the blue curve for SA1 and the orange (or red curves) are identical. The next two tests compare the DFs of LA1 and IA1 integrated in the correct potential (blue curves) with the CDFs when the same ensembles are evolved in the potential for SA1 (orange curves). These two tests also yielded small values of  $p$  implying a low probability that the CDFs of LA1 and IA1 were identical to those of LA1-in-SA1 and IA1-in-SA1, respectively. Thus, the first five KS tests show that the probability of CDFs of  $\log(\Delta f)$  arising from the correct and incorrect potentials being identical is very low ( $p \sim 0$ ).

The next three tests compared various ‘slightly incorrect distributions’ (orange curves in Fig. 12) with each other. The results of the KS tests in Table 2 show that it is difficult to statistically discriminate between the various slightly incorrect potentials because the probabilities of their being identical are quite large  $p > 0.1$ .

The last two tests compared two ‘slightly incorrect’ models (orange curves) with the ‘strongly incorrect’ models (red curve) (labelled ‘Halo A + DISC’): the small values of  $p$  and  $d$  indicate that the red curve can be statistically distinguished from the orange curves with high confidence.

The above tests show that although each particle is treated as a ‘test-particle’ in the fixed background potential we are able to harness the collective behaviour of an ensemble *drawn from a self-consistent distribution function assumed to be in steady state* to statistically identify cases where the orbit ensemble is not in equilibrium with the background potential. Although the diffusion rate of an ensemble (in a fixed background potential) can be quantified even without self-consistent calculations, it is more correct to think of this diffusion in terms of the collective relaxation that occurs when an  $N$ -body system is out of equilibrium, rather than chaos, since these orbits are drawn from a DF.

This is a fundamental result that relies on the Jeans Theorem: when the initial positions and velocities (phase-space coordinates) of orbits are not drawn from a self-consistent DF in a steady-state equilibrium potential, most orbits are not launched on regular tori and hence they will diffuse in phase space (travel through the ‘Arnold web’; cf. Lichtenberg & Lieberman 1992). The results given above show that although there is a spread of diffusion rates in any given ensemble. Since the CDF of the distribution for the entire ensemble is shifted to smaller values of the diffusion rate when the potential is close to equilibrium and becomes statistically larger when the potential is incorrect.

From Fig. 12 we see that the CDF of diffusion rates of a ‘slightly incorrect’ model can be clearly distinguished from the correct model. This suggests a novel way to utilize the six phase-space coordinates of MW halo stars to distinguish between various possible models for the halo potential. However, the overlap of orange curves for various ‘slightly incorrect’ models indicates that it may not be easy to distinguish between the various ‘slightly incorrect’ models in a way that allows us to progress iteratively towards a model closest to the true potential.

None the less this method has the value that it will work best in the region of the MW halo that is more well mixed (i.e. the inner halo), whereas methods such as modelling the tidal tails of dwarf satellites relies on their being less well mixed and will primarily be applicable in the outer halo. Note that it is not necessary to assume that all the orbits in the correct equilibrium model are regular. In fact, the CDF of diffusion rates in Fig. 12 shows that even in the correct/equilibrium models there are a few orbits which have fairly high diffusion rates.

It is important to note that this method will not work if the net potential is spherical (which the MW is not because  $\sim 10$  per cent of the total mass is in the highly flattened Galactic disc), or if the potential is assumed to be of Stäckel form. This is because all orbits in spherical and Stäckel potentials are integrable by definition (i.e. all orbits are perfectly regular), hence the diffusion of orbits would only measure numerical errors of the method. However, since Stäckel potentials are idealized potentials of primarily theoretical interest, there is no reason to expect that the Galaxy should have such a form. In a future paper we will refine these ideas to quantitatively assess the possibility of measuring the shape of the MW’s inner halo with ensembles of stellar orbits.

## 5 DISCUSSION AND CONCLUSIONS

Understanding the structure, dynamics and formation history of the MW is a major thrust of current astronomical research. The fundamental science driver of the burgeoning Galactic all-sky survey industry, e.g. SDSS-SEGUE (Yanny et al. 2009), Apache Point Observatory Galactic Evolution Experiment (APOGEE; Allende Prieto et al. 2008), RAVE (Steinmetz 2003), (Large Synoptic Survey Telescope (LSST; LSST Science Collaborations 2009), Panoramic

Survey Telescope & Rapid Response System (PanSTARRS; Kaiser et al. 2002), Large Sky Area Multi-Object Fiber Spectroscopic Telescope (LAMOST; Hu & Jiang 2005), SkyMapper (Keller et al. 2007), High Efficiency and Resolution Multi-Element Spectrograph (HERMES; Barden et al. 2008) and eventually *Gaia* (Perryman et al. 2001), is the premise that the Galaxy and the Local Group were formed in a manner that typifies the formation of galaxies in the  $\Lambda$ CDM paradigm. Therefore understanding the structure, dynamics, chemistry and thereby the formation history of our own Galaxy will result in stronger constraints on the complex physics of galaxy formation.

The use of orbital frequencies of halo stars has been recognized as an important way of identifying the relics of satellites which were tidally disrupted in the MW halo (McMillan & Binney 2008; Gómez & Helmi 2010). In a recent paper, Gómez et al. (2010) also showed that when the time-dependent growth of the Galaxy and observational errors associated with *Gaia* are taken into account, it will be possible to uniquely identify about 30 per cent of the accretion events that occurred in the last 10 Gyr, while the remainder of the accretion events will likely be difficult to disentangle from a more smoothly mixed component. Helmi et al. (2011) have argued, from a comparison of the degree of spatial variation in the distribution of stars in the stellar halo observed in the SDSS-II survey (Bell et al. 2008) and the similar distributions of halo stars in cosmological  $N$ -body simulations (Cooper et al. 2010), that there may exist a smooth underlying stellar halo component that is much more well mixed than material solely accreted from satellites. The method outlined here complements the work that focuses on disentangling the relics of tidal streams, in that it can be applied to both well-mixed and unmixed orbits and can therefore be applied to a much larger sample of halo stellar orbits.

The SDSS-SEGUE survey has already obtained phase-space coordinates for over 17 000 stars within 4 kpc of the Sun. We are applying the techniques described in this paper to the SDSS-Segue Calibration sample (Carollo et al. 2010), and correlate the orbital properties of halo stars with their metallicities and spatial distribution (Valluri 2011; O’Shea et al., in preparation) use revised distances to the stars in the Calibration sample (Beers et al. 2011) that overcome recently reported distance measurement errors (Schönrich et al. 2011). Upcoming surveys will increase both the volume of space observed and the accuracy with which distances, proper-motions, radial velocities and metallicities are measured for MW halo stars, significantly impacting our understanding of the structure and dynamics of the Galaxy. If the stellar halo does consist of distinct components – a well-mixed inner halo which formed largely *in situ* (i.e. in potentials deeper than typical dwarf satellite potentials) and an outer halo that was formed primarily from the accretion of tidally stripped dwarf satellites – one might also expect to observe distinct differences in the orbital populations of these two components.

One of the primary goals of current and future surveys of the MW is to determine the shape and radial density profile of the dark matter halo and to construct the self-consistent phase-space DF of the major stellar components – the thin and thick discs, the bulge and the stellar halo (Binney & McMillan 2011). A popular and flexible method for constructing the DF of a potential is the orbit superposition method (Schwarzschild 1979; van der Marel et al. 1998; Cretton et al. 1999; Thomas et al. 2004; Valluri, Merritt & Emsellem 2004), which relies on orbit libraries that uniformly sample orbital initial condition space. In the DF of realistic triaxial potentials like the haloes of galaxies, a large fraction of box-like orbits are resonant (i.e. have commensurable frequencies) (Miralda-Escude & Schwarzschild 1989; Merritt & Valluri 1999). Determining the frac-

tion of orbits associated with resonances is therefore important for constructing the DF. However, since these orbits are ‘resonantly trapped’, they densely populate very small regions of phase space, and are likely to be underrepresented in the orbit libraries used to construct DFs (which generally sample some initial condition space uniformly). Resonant orbits are also not adequately represented in the DFs constructed via orbital torus construction methods, since they are, by design, not present in perfectly regular Stäckel potentials, whose orbital tori are adiabatically deformed to construct the DF of realistic potentials (e.g. Binney 2010; Binney & McMillan 2011). The methods described in this paper address this important issue in a uniquely powerful way. Rather than attempting to guess the initial distribution of orbits required to populate the orbit libraries, the frequency analysis of the orbits of halo stars in a set of trial potentials can be used to construct the frequency map, which we have shown to yield a robust picture of the global DF, even when orbits are selected within a limited volume in the Galaxy. The measurement of the diffusion rates of the orbits then gives an estimate of the best trial potential, and can be used as an additional constraint in the Schwarzschild or torus construction method. The applicability of these methods to the construction of the full phase-space DF of the stellar halo will be discussed in greater detail in future work.

In the majority of simulations presented in this paper we do not consider the possibility that the inner and outer haloes could have angular momenta of different directions or signs. Warps seen in disc galaxies are now believed to be evidence that the angular momentum vectors of outer dark matter halo is misaligned from the disc and the associated inner halo (e.g. Ostriker & Binney 1989; Debattista & Sellwood 1999; Jiang & Binney 1999; Shen & Sellwood 2006; Roškar et al. 2010). These authors argue that cosmic infall causes the angular momentum direction of the outer dark matter halo to be different from that of the inner dark matter halo. The build up of dark matter haloes via cosmic infall occurs due to the accretion of satellites, and such hierarchical infall is also believed to build up the stellar halo. In a companion paper (Valluri et al., in preparation) we study the angular momentum distributions of both star particles and dark matter particles in high-resolution cosmological simulations of disc galaxies from the MUGS collaboration (Stinson et al. 2010). Preliminary results indicate that the angular momentum distributions of halo stars at different radial distances from the galactic centre can be used to trace the angular momentum distribution of the dark matter haloes at these radii. Thus the orbital analysis of halo stars could potentially be used to measure the angular momentum of the dark matter halo and test the prediction that differential angular momentum of the halo can produce warps. Model TA1 [Figs 4 (right-hand panel) and 7] shows the effect of a disc inclined to the equatorial plane of a triaxial halo. Fig. 4 (right-hand panel) shows that the density contours of the inner halo become aligned with the disc, while the outer halo remains unchanged. In this simulation the disc was held rigid, but it is likely that a live disc would form a warp due to the misalignment of the inner and outer halo angular momenta. None the less this simulation is illustrative of the possibility of using frequency maps of halo stars to detect such tilts in the principal axes and angular momentum directions of different parts of the halo since it illustrates that the orbital frequencies only depend on the assumed potential.

To conclude we summarize the main results of this paper.

- (i) Frequency analysis of the orbits of a large representative sample of halo particles can be used to construct frequency maps which provide a compact representation of the DF of the stellar or dark



matter halo. The map also gives an estimate of the fractions of orbits in different major orbit families and enables easy identification of dominant resonances. The identification of global resonances is important for constructing global DFs for the MW since global resonances strongly constrain the DF. The ability of frequency maps to reveal globally important orbit families and resonances, even with orbits selected from a limited volume around the Sun, suggests that this method will provide significant input to the construction of DFs for the MW Galaxy (May & Binney 1986).

(ii) The adiabatic growth of a disc in a halo results in significant resonant trapping of halo particles with resonant orbits appearing clustered along narrow resonance lines on a frequency map. In a Cartesian frequency map of orbits in a moderately triaxial halo we see several resonant box-orbit families. In axisymmetric haloes, the use of cylindrical coordinates reveals a large number of resonances primarily between the radial frequency  $\Omega_R$  and the vertical frequency  $\Omega_z$ .

(iii) Resonances are found in all of the controlled simulations in which discs were grown inside haloes, regardless of the shape of the halo and regardless of how the disc was oriented relative to the large-scale orientation of the halo. It has been previously demonstrated that the growth of a baryonic disc inside a triaxial halo deforms the inner regions to make them more oblate, but the haloes remain modestly triaxial at large radii. We find that although the inner regions of the halo are similar regardless of the large-scale halo orientation, the frequency maps of inner halo particles reflect the differences in the global orientation of the halo relative to the disc.

(iv) We see that halo resonances are formed by both static discs and live discs (those that form spiral features and bars). However, the coherent time-dependent perturbations from a bar can result in scattering of the most tightly bound particles, resulting in broader (less well defined) resonances.

(v) Controlled hydrodynamic simulations, in which hot gas distributed throughout a dark matter halo is allowed to cool and form a gas disc which then forms stars, give rise to halo frequency maps with resonant structure, much like the adiabatic simulations. This shows that resonant trapping of halo stars by the disc is not purely a feature of idealized collisionless simulations in which a rigid disc is grown in place.

(vi) We find that the cumulative distribution of orbital diffusion rates is lower by a statistically significant amount, when a large ensemble of orbits is integrated in a potential in which it is in self-consistent equilibrium, and that the orbital diffusion rates are significantly larger when the potential is incorrect. This can potentially provide a novel way to constrain the potential of the MW directly from the six phase-space coordinates of a large sample of MW halo stars.

## ACKNOWLEDGMENTS

MV is supported by NSF grant AST-0908346. We thank Greg Stinson and Jeremy Bailin for detailed comments on an earlier version of this manuscript. MV thanks Tim Beers, Eric Bell and Mario Mateo for enlightening discussions on SEGUE and other resolved-star all-sky survey data.

## REFERENCES

Abadi M. G., Navarro J. F., Steinmetz M., Eke V. R., 2003, *ApJ*, 591, 499  
 Allende Prieto C. et al., 2008, *Astron. Nachr.*, 329, 1018  
 Allgood B., Flores R. A., Primack J. R., Kravtsov A. V., Wechsler R. H., Faltenbacher A., Bullock J. S., 2006, *MNRAS*, 367, 1781

Athanassoula E., 2002, *ApJ*, 569, L83  
 Bailin J., Steinmetz M., 2005, *ApJ*, 627, 647  
 Barden S. C. et al., 2008, in McLean I. S., Casali M. M., eds, *Proc. SPIE Conf. Ser. Vol. 7014, Ground-based and Airborne Instrumentation for Astronomy II*. SPIE, Bellingham, p. 70144J  
 Battaglia G. et al., 2005, *MNRAS*, 364, 433  
 Beers T. C. et al., 2011, preprint (arXiv e-prints)  
 Bell E. F., Zucker D. B., co authors, 2008, *ApJ*, 680, 295  
 Belokurov V. et al., 2006, *ApJ*, 642, L137  
 Belokurov V. et al., 2007, *ApJ*, 654, 897  
 Berentzen I., Shlosman I., 2006, *ApJ*, 648, 807  
 Binney J., 2010, *MNRAS*, 401, 2318  
 Binney J., McMillan P., 2011, *MNRAS*, 413, 1889  
 Binney J., Spergel D., 1982, *ApJ*, 252, 308  
 Binney J., Spergel D., 1984, *MNRAS*, 206, 159  
 Binney J., Tremaine S., 2008, *Galactic Dynamics*, 2nd edn. Princeton Univ. Press, Princeton, NJ  
 Bullock J. S., Johnston K. V., 2005, *ApJ*, 635, 931  
 Bullock J. S., Kolatt T. S., Sigad Y., Somerville R. S., Kravtsov A. V., Klypin A. A., Primack J. R., Dekel A., 2001, *MNRAS*, 321, 559  
 Carollo D. et al., 2007, *Nat*, 450, 1020  
 Carollo D. et al., 2010, *ApJ*, 712, 692  
 Carpintero D. D., Aguilar L. A., 1998, *MNRAS*, 298, 1  
 Choi J., Weinberg M. D., Katz N., 2009, *MNRAS*, 400, 1247  
 Cooper A. P. et al., 2010, *MNRAS*, 406, 744  
 Cretton N., de Zeeuw P. T., van der Marel R. P., Rix H.-W., 1999, *ApJS*, 124, 383  
 De Lucia G., Helmi A., 2008, *MNRAS*, 391, 14  
 de Zeeuw P. T., Hunter C., 1990, *ApJ*, 356, 365  
 Debattista V. P., Sellwood J. A., 1999, *ApJ*, 513, L107  
 Debattista V. P., Sellwood J. A., 2000, *ApJ*, 543, 704  
 Debattista V. P., Moore B., Quinn T., Kazantzidis S., Maas R., Mayer L., Read J., Stadel J., 2008, *ApJ*, 681, 1076 (D08)  
 Deibel A. T., Valluri M., Merritt D., 2011, *ApJ*, 728, 128  
 Dierickx M., Klement R., Rix H., Liu C., 2010, *ApJ*, 725, L186  
 Dubinski J., Carlberg R. G., 1991, *ApJ*, 378, 496  
 Font A. S., Johnston K. V., Ferguson A. M. N., Bullock J. S., Robertson B. E., Tumlinson J., Guhathakurta P., 2008, *ApJ*, 673, 215  
 Gnedin O. Y., Gould A., Miralda Escudé J., Zentner A. R., 2005, *ApJ*, 634, 344  
 Gómez F. A., Helmi A., 2010, *MNRAS*, 401, 2285  
 Gómez F. A., Helmi A., Brown A. G. A., Li Y., 2010, *MNRAS*, 408, 935  
 Heiligman G., Schwarzschild M., 1979, *ApJ*, 233, 872  
 Helmi A., 2004, *MNRAS*, 351, 643  
 Helmi A., de Zeeuw P. T., 2000, *MNRAS*, 319, 657  
 Helmi A., Cooper A. P., White S. D. M., Cole S., Frenk C. S., Navarro J. F., 2011, *ApJ*, 733, L7  
 Hoffman L., Cox T. J., Dutta S., Hernquist L., 2009, *ApJ*, 705, 920  
 Hoffman L., Cox T. J., Dutta S., Hernquist L., 2010, *ApJ*, 723, 818  
 Hu J., Jiang B., 2005, in Mikolajewska J., Olech A., eds, *AIP Conf. Ser. Vol. 752, Stellar Astrophysics with the World's Largest Telescopes*. Am. Inst. Phys., New York, p. 84  
 Jiang I., Binney J., 1999, *MNRAS*, 303, L7  
 Jing Y. P., Suto Y., 2002, *ApJ*, 574, 538  
 Johnston K. V., Hernquist L., Bolte M., 1996, *ApJ*, 465, 278  
 Johnston K. V., Zhao H., Spergel D. N., Hernquist L., 1999, *ApJ*, 512, L109  
 Johnston K. V., Bullock J. S., Sharma S., Font A., Robertson B. E., Leitner S. N., 2008, *ApJ*, 689, 936  
 Kaiser N. et al., 2002, in Tyson J. A., Wolff S., eds, *Proc. SPIE Conf. Ser. Vol. 4836, Pan-STARRS: A Large Synoptic Survey Telescope Array*. SPIE, Bellingham, p. 154  
 Kalapotharakos C., 2008, *MNRAS*, 389, 1709  
 Kandrup H. E., Sideris I. V., 2003, *ApJ*, 585, 244  
 Katz N., 1991, *ApJ*, 368, 325  
 Kazantzidis S., Kravtsov A. V., Zentner A. R., Allgood B., Nagai D., Moore B., 2004, *ApJ*, 611, L73



- Kazantzidis S., Bullock J. S., Zentner A. R., Kravtsov A. V., Moustakas L. A., 2008, *ApJ*, 688, 254
- Kazantzidis S., Abadi M. G., Navarro J. F., 2010, *ApJ*, 720, L62
- Keller S. C. et al., 2007, *Publ. Astron. Soc. Australia*, 24, 1
- Laskar J., 1990, *Icarus*, 88, 266
- Laskar J., 1993, *Celest. Mech. Dynamical Astron.*, 56, 191
- Laskar J., Froeschlé C., Celletti A., 1992, *Phys. D: Nonlinear Phenomena*, 56, 253
- Law D. R., Majewski S. R., 2010, *ApJ*, 714, 229
- Law D. R., Majewski S. R., Johnston K. V., 2009, *ApJ*, 703, L67
- Lichtenberg A. J., Lieberman M. A., 1992, *Regular and Chaotic Motion*. Applied Mathematical Sciences. Springer, New York
- Loebman S. R., Roskar R., Debattista V. P., Ivezić Z., Quinn T. R., Wadsley J., 2010, *ApJ*, 737, 8
- LSST Science Collaborations and LSST Project 2009, *LSST Science Book*, Version 2.0, preprint (arXiv:0912.0201), <http://www.lsst.org/lsst/scibook>
- McMillan P. J., Binney J. J., 2008, *MNRAS*, 390, 429
- Malhotra R., 1993, *Nat*, 365, 819
- May A., Binney J., 1986, *MNRAS*, 221, 857
- Merritt D., Valluri M., 1996, *ApJ*, 471, 82
- Merritt D., Valluri M., 1999, *AJ*, 118, 1177
- Miralda Escude J., Schwarzschild M., 1989, *ApJ*, 339, 752
- Navarro J. F., Frenk C. S., White S. D. M., 1996, *ApJ*, 462, 563
- Newberg H. J. et al., 2002, *ApJ*, 569, 245
- Ostriker E. C., Binney J. J., 1989, *MNRAS*, 237, 785
- Papaphilippou Y., Laskar J., 1996, *A&A*, 307, 427
- Perryman M. A. C. et al., 2001, *A&A*, 369, 339
- Poon M. Y., Merritt D., 2004, *ApJ*, 606, 774
- Quillen A. C., 2006, *MNRAS*, 365, 1367
- Robutel P., Laskar J., 2001, *Icarus*, 152, 4
- Roškar R., Debattista V. P., Stinson G. S., Quinn T. R., Kaufmann T., Wadsley J., 2008, *ApJ*, 675, L65
- Roškar R., Debattista V. P., Brooks A. M., Quinn T. R., Brook C. B., Governato F., Dalcanton J. J., Wadsley J., 2010, *MNRAS*, 408, 783
- Sales L. V. et al., 2009, *MNRAS*, 400, L61
- Schönrich R., Asplund M., Casagrande L., 2011, *MNRAS*, 415, 3807
- Schönrich R., Binney J., 2009, *MNRAS*, 399, 1145
- Schwarzschild M., 1979, *ApJ*, 232, 236
- Shen J., Sellwood J. A., 2006, *MNRAS*, 370, 2
- Sridhar S., Touma J., 1996, *MNRAS*, 279, 1263
- Stadel J. G., 2001, PhD thesis, Univ. Washington
- Steinmetz M., 2003, in Munari U., ed., *ASP Conf. Ser. Vol. 298, GAIA Spectroscopy: Science and Technology*. Astron. Soc. Pac., San Francisco, p. 381
- Steinmetz M. et al., 2006, *AJ*, 132, 1645
- Stinson G., Seth A., Katz N., Wadsley J., Governato F., Quinn T., 2006, *MNRAS*, 373, 1074
- Stinson G. S., Bailin J., Couchman H., Wadsley J., Shen S., Nickerson S., Brook C., Quinn T., 2010, *MNRAS*, 408, 812
- Thomas J., Saglia R. P., Bender R., Thomas D., Gebhardt K., Magorrian J., Richstone D., 2004, *MNRAS*, 353, 391
- Tissera P. B., White S. D. M., Pedrosa S., Scannapieco C., 2010, *MNRAS*, 406, 922
- Tremaine S., Weinberg M. D., 1984, *MNRAS*, 209, 729
- Tremaine S., Yu Q., 2000, *MNRAS*, 319, 1
- Valluri M., 2011, preprint (arXiv:1110.2486)
- Valluri M., Merritt D., 1998, *ApJ*, 506, 686
- Valluri M., Merritt D., Emsellem E., 2004, *ApJ*, 602, 66
- Valluri M., Vass I. M., Kazantzidis S., Kravtsov A. V., Bohn C. L., 2007, *ApJ*, 658, 731
- Valluri M., Debattista V. P., Quinn T., Moore B., 2010, *MNRAS*, 403, 525 (V10)
- van den Bosch F. C., Abel T., Croft R. A. C., Hernquist L., White S. D. M., 2002, *ApJ*, 576, 21
- van der Marel R. P., Cretton N., de Zeeuw P. T., Rix H., 1998, *ApJ*, 493, 613
- Villalobos Á., Helmi A., 2008, *MNRAS*, 391, 1806
- Wadsley J. W., Stadel J., Quinn T., 2004, *New Astron.*, 9, 137
- Weinberg M. D., 1985, *MNRAS*, 213, 451
- Willman B. et al., 2005, *ApJ*, 626, L85
- Wilson M. et al., 2011, *MNRAS*, 413, 2235
- Yanny B. et al., 2003, *ApJ*, 588, 824
- Yanny B., Rockosi C., Newberg H. J., Knapp G. R., the SDSS Collaboration, 2009, *AJ*, 137, 4377
- Yoachim P., Dalcanton J. J., 2008, *ApJ*, 682, 1004
- Yu Q., Tremaine S., 2001, *AJ*, 121, 1736
- Zentner A. R., Kravtsov A. V., Gnedin O. Y., Klypin A. A., 2005, *ApJ*, 629, 219
- Zucker D. B. et al., 2006, *ApJ*, 643, L103

This paper has been typeset from a  $\text{\LaTeX}$  file prepared by the author.