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A recursive formula for the product of element orders of finite abelian groups

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Abstract. Let G be a finite group and let $\psi(G)$ denote the sum of element orders of G; later this concept has been used to define $R(G)$ which is the product of the element orders of G. Motivated by the recursive formula for $\psi(G)$, we consider a finite abelian group G and obtain a similar formula for $R(G)$.

1 Introduction

Let G be a finite group. For any non-empty subset S of G, let $\psi(S)$ denote the sum of element orders of S. This has been introduced in [2] and later in [4], the notion $R(G)$ was introduced which stands for the product of element orders of G. In the same paper, a formula for computing $R(G)$ when G is a finite abelian group was obtained. In [3], [5], an explicit recursive formula for computing $\psi(G)$ were obtained in case G is abelian. Motivated by these results, in this paper, we obtain a similar recursive formula for computing $R(G)$ when G is a finite abelian group.

Throughout this paper, we let $\varphi(n)$ denote the Euler totient function of the positive integer n and let p denote a prime number. A cyclic group of order n will be denoted by C_n whereas $C_p^{(r)}$ will denote the elementary abelian p -group of rank r. We always assume G to be finite. For a group G and element $x \in G$, the notation $o(x)$ denotes the order of x. For any group G, we take

$$R(G) = \prod_{x \in G} o(x).$$

For a group G, the notation $\exp(G)$ denotes the exponent of G which is the smallest positive integer z such that $g^z = 1_G$ for all $g \in G$ where 1_G is the identity element of G; without any ambiguity we will denote this identity element as 1.

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2 Explicit formulas for finite abelian groups

In this section, we obtain explicit recursive formula for $R(G)$ where G is a finite abelian group. We present this in different cases starting from a finite cyclic group. This format is inspired by [3, Section 2]. We will then consider the direct product of a finite cyclic p-group and a (not necessarily abelian) p-group. Finally, we will consider a the most general case of finite abelian groups. The proofs of our results in this section are motivated by the methods used in [3]. We begin with the following important preliminary results.

Theorem 2.1 ([4, Proposition 1.1]). *Let $G_1, G_2, \ldots, G_k$ be finite groups having co-prime orders and $G \cong G_1 \times G_2 \times \cdots \times G_k$. Then*

$$R(G) = \prod_{i=1}^{k} R(G_i)^{n_i}, \quad \textit{where} \quad n_i = \prod_{j=1, j\neq i}^{k} |G_j|, i = 1, 2, \ldots, k.$$

Lemma 2.2 ([3, Lemma 2.6]). *Let $H \cong C_{p^{r_1}} \times C_{p^{r_2}} \times \cdots \times C_{p^{r_n}}$ where $1 \leq r_1 \leq r_j$ for all j with $2 \leq j \leq n$. Then for any $i \in \{1, \ldots, r_1\}$, there are $(p^i)^n - (p^{i-1})^n$ elements of H of order p^i*

The following lemma, motivated by [3, Lemma 1.1], follows easily from the fact that C_n has exactly $\varphi(d)$ elements of order d for each divisor d of n.

Lemma 2.3. *Let n be any positive integer. Then*

$$\psi(C_n) = \sum_{d|n} d\varphi(d) \quad \text{and} \quad R(C_n) = \prod_{d|n} d^{\varphi(d)}$$

.

2.1 Finite Cyclic Groups

Let G be a cyclic group of order n. Then we know that $G \cong C_{m_1} \times \cdots \times C_{m_k}$, where $m_1, \ldots, m_k$ are co-prime to each other and $n = m_1 \ldots m_k$.

Lemma 2.4. *Let G be a cyclic group of order p^n where p is a prime number and n is a positive integer, then $R(G) = p^{\left(\frac{p+np^{n+2}-(n+1)p^{n+1}}{p(p-1)}\right)}$.*

Proof. Using Lemma 2.3, we get $R(G) = \prod_{r=1}^{n} p^{\varphi(p^r)r} = p^{\sum_{r=1}^{n} \varphi(p^r)r}$. So we have $R(G) = p^z$ where $z = \sum_{r=1}^{n} r(p^r - p^{r-1}) = (1 - \frac{1}{p}) \sum_{r=1}^{n} rp^r$. Now the sum $\sum_{r=1}^{n} rp^r$ is an arithmetic-geometric series with common difference 1 and common ration p. Thus

$$\sum_{r=1}^{n} rp^r = \frac{p + np^{n+2} - (n+1)p^{n+1}}{(p-1)^2}.$$

This shows that $z = \frac{p+np^{n+2}-(n+1)p^{n+1}}{p(p-1)}$. So we get $R(G) = p^{\left(\frac{p+np^{n+2}-(n+1)p^{n+1}}{p(p-1)}\right)}$. ∎

Lemma 2.5. *Let G be a cyclic group of order $s = p_1^{r_1} \dots p_k^{r_k}$ where the p_i are distinct primes with $r_i \geq 1$ for $i = 1, \dots, k$. Then*

$$R(G) = \prod_{i=1}^{k} p_i^{\left(\frac{p_i + r_i p_i^{r_i+2} - (r_i+1) p_i^{r_i+1}}{p_i(p_i-1)}\right) n_i}, \quad n_i = p_i^{-r_i} \prod_{j=1}^{k} p_j^{r_j}.$$

Proof. We know that $G \cong C_{p_1^{r_1}} \times \cdots \times C_{p_k^{r_k}}$ and p_i are all distinct primes. Hence we can apply Theorem 2.1. Thus by Lemma 2.4, we arrive at the required result. $\square$

2.2 Direct product of a cyclic p-Group and a p-Group

In this section, we obtain a recursive formula for $R(G)$ where G is a direct product of a finite cyclic p-group and any p-group. We begin with the following lemma from [3, Lemma 2.3].

Lemma 2.6. *If H and K are p-groups, then $o((x_1, x_2)) = \max\{o(x_1), o(x_2)\}$ for all for any $x_1 \in H$ and $x_2 \in K$.*

We now prove the following using techniques motivated by [3].

Proposition 2.7. *Let $G = C_{p^r} \times H$ where $r \geq 1$, and H is a p-group with $\exp(H) \geq p^r$. Let N_j be the number of elements in H that have order p^j. Then*

$$R(G) = \begin{cases} p^{p-1} R(H)^{p^r} \prod_{i=2}^{r} \left(\prod_{j=1}^{i-1} (p^{i-j})^{N_j}\right)^{p^i - p^{i-1}}, & \text{if } r > 1 \\ p^{p-1} R(H)^p, & \text{if } r = 1 \end{cases}$$

Proof. Note that G is a finite group whose elements are of the form (x, y) where $x \in C_{p^r}$ and $y \in H$. We now partition G based on the order of the elements in the first component. In particular we have, $G = \bigcup_{k=0}^{r} F_k$ where $F_k = \left\{(x_1, x_2) \in G \mid o(x_1) = p^k\right\}$. Since the $F_i \cap F_j = \emptyset$ for $i \neq j$, we have $R(G) = \prod_{k=0}^{r} R(F_k)$. Now let $x_1 \in C_{p^r}$ with $o(x_1) = p^i$ for some i with $0 \leq i \leq r$. For each such x_1, define $F_{i,x_1} = \{(x_1, x_2) \mid x_2 \in H\}$. Then we have $F_i = \bigcup_{x_1 \in C_{p^r}, o(x_1) = p^i} F_{i,x_1}$ and $R(F_i) = \prod_{x_1 \in C_{p^r}, o(x_1) = p^i} R(F_{i,x_1})$ for $i = 0, 1, \dots, r$. There are $(p^i - p^{i-1})$ elements of order p^i in C_{p^r}, see Lemma 2.2. As a result,

$$R(F_i) = R(F_{i,x_1})^{(p^i - p^{i-1})}.$$

Taking $i = 0$, we have $F_0 = F_{0,1}$ and thus, $R(F_0) = R(H)$. For $i = 1$, each element (x_1, x_2) in F_{1,x_1} has order same as $o(x_2)$ except for $(x_1, 1)$ which has order p. Thus $R(F_{1,x_1}) = pR(H)$. So $R(F_1) = (pR(H))^{p-1}$.

If $r = 1$ then $R(G) = R(F_0) R(F_1) = p^{p-1} R(H)^p$.

Now consider $r > 1$ and let $i \in \{2, \dots, r\}$. For $(x_1, x_2) \in F_{i,x_1}$ with $o(x_2) = p^j$, we have

$o((x_1, x_2)) = p^i$ if $j < i$ and $o((x_1, x_2)) = p^j$ if $j \geq i$. If $r' = \exp(H)$ then

$$
\begin{aligned}
R\left(F_{i,x_1}\right) &= \prod_{j=0}^{i-1} (p^i)^{N_j} \prod_{j=i}^{r'} (p^j)^{N_j} \\
&= p^i \prod_{j=1}^{i-1} (p^i)^{N_j} \prod_{j=i}^{r'} (p^j)^{N_j} \\
&= p^i \prod_{j=1}^{r'} (p^j)^{N_j} \prod_{j=1}^{i-1} (p^{i-j})^{N_j} \\
&= p^i R(H) \prod_{j=1}^{i-1} (p^{i-j})^{N_j}
\end{aligned}
$$

Thus $\prod_{i=2}^{r} R\left(F_i\right) = R(H)^{(p^r - p)} \prod_{i=2}^{r} \left(\prod_{j=1}^{i-1} (p^{i-j})^{N_j}\right)^{p^i - p^{i-1}}$ and finally the result follows from a direct calculation $R(G) = R\left(F_0\right) R_l\left(F_1\right) \prod_{i=2}^{r} R_l\left(F_i\right)$. □

2.3 Finite abelian groups

We can now state how to compute $R(G)$ for any finite abelian group G. In view of Theorem 2.1, the following is a direct application of Proposition 2.7 and Lemma 2.2.

Theorem 2.8. *Let G be a finite abelian group with $G \cong H_1 \times \cdots \times H_k$ where each H_i is an abelian p_i-group and p_i are distinct primes for $i = 1, \ldots, k$. Then*

$$R(G) = R\left(H_1\right) \ldots R\left(H_k\right)$$

where $R\left(H_i\right)$ for $i = 1, \ldots, k$ are computed as follows:

a) *If $H_i \cong C_{p_i^n}$ then*

$$R(H_i) = p_i^{\left(\frac{p_i + n p_i^{n+2} - (n+1) p_i^{n+1}}{p_i (p_i - 1)}\right)}.$$

b) *If $H_i \cong C_{p_i^{r_1}} \times C_{p_i^{r_2}} \times \cdots \times C_{p_i^{r_n}}$, where $1 \leq r_1 \leq r_2 \leq \cdots \leq r_n$, and $r_1 + \ldots + r_n = r$ then $R\left(H_i\right)$ can be determined recursively as follows*

i) *If $r_1 > 1$ then*

$$R(H_i) = p_i^{p_i - 1} R(C_{p_i^{r_2}} \times \cdots \times C_{p_i^{r_n}})^{p^{r_1}} \prod_{z=2}^{r_1} \left(\prod_{j=1}^{z-1} (p_i^{z-j})^{N_j}\right)^{p_i^z - p_i^{z-1}},$$

where $N_j = \left(\left(p_i^j\right)^{n-1} - \left(p_i^{j-1}\right)^{n-1}\right)$

ii) *If $r_1 = 1$ then*

$$R(H_i) = p_i^{p_i - 1} R(C_{p_i^{r_2}} \times \cdots \times C_{p_i^{r_n}})^{p_i}.$$

3 Some Examples

In this final section we compute some examples using Theorem 2.8.

Example 3.1. We compute $R(C_p^{(r)} \times C_{p^n})$ where n and r are positive integers. By part a) of Theorem 2.8 we have

$$R(C_{p^n}) = p^{\left(\frac{p+np^{n+2}-(n+1)p^{n+1}}{p(p-1)}\right)}.$$

Then by part b) of Theorem 2.8 we have

$$R(C_p \times C_{p^n}) = p^{p-1}p^{p\left(\frac{p+np^{n+2}-(n+1)p^{n+1}}{p(p-1)}\right)} = p^{\left(\frac{(p-1)^2+(p+np^{n+2}-(n+1)p^{n+1})}{(p-1)}\right)}.$$

Similarly

$$R(C_p \times C_p \times C_{p^n}) = p^{\left(\frac{(p-1)^2(1+p)+p(p+np^{n+2}-(n+1)p^{n+1})}{(p-1)}\right)}.$$

Thus inductively it is easy to show that

$$R(C_p^{(r)} \times C_{p^n}) = p^{\left(\frac{(p-1)^2(1+p+\ldots+p^{r-1})+p^{r-1}(p+np^{n+2}-(n+1)p^{n+1})}{(p-1)}\right)}.$$

Thus we have

$$R(C_p^{(r)} \times C_{p^n}) = p^{\left(\frac{(p-1)(p^r-1)+p^{r-1}(p+np^{n+2}-(n+1)p^{n+1})}{(p-1)}\right)}.$$

The next example is an application of Theorem 2.8 in the case where $r_1 > 1$.

Example 3.2. In this example we compute $R(C_{p^2} \times C_{p^n})$ where n is a positive integers. By part a) of Theorem 2.8 we have

$$R(C_{p^n}) = p^{\left(\frac{p+np^{n+2}-(n+1)p^{n+1}}{p(p-1)}\right)}.$$

Then by part b) of Theorem 2.8 we have

$$R(C_{p^2} \times C_{p^n}) = p^{p-1}p^{p^2\left(\frac{p+np^{n+2}-(n+1)p^{n+1}}{p(p-1)}\right)}p^{p(p-1)(p^{n-1}-1)}.$$

This gives

$$R(C_{p^2} \times C_{p^n}) = p^{\left(\frac{(p-1)^2(p^n+1-p)+p(p+np^{n+2}-(n+1)p^{n+1})}{p-1}\right)}.$$

In the following example we compute $R(C_p^{(r)} \times C_{p^2} \times C_{p^n})$ where n and r are positive integers with $n \geq 2$.

Example 3.3. In example 3.2 we have computed $R(C_{p^2} \times C_{p^n})$. Then by part b) of Theorem 2.8 we have

$$R(C_p \times C_{p^2} \times C_{p^n}) = p^{p-1}p^{p\left(\frac{(p-1)^2(p^n+1-p)+p(p+np^{n+2}-(n+1)p^{n+1})}{p-1})\right)}.$$

This gives

$$R(C_p \times C_{p^2} \times C_{p^n}) = p^{p\left(\frac{(p-1)^2(1+p-p^2+p^{n+1})+p^2(p+np^{n+2}-(n+1)p^{n+1})}{p-1})\right)}.$$

Then inductively one can show that

$$R(C_p^{(r)} \times C_{p^2} \times C_{p^n}) = p^{p\left(\frac{(p-1)^2(1+p+\ldots+p^r-p^{r+1}+p^{n+r})+p^{r+1}(p+np^{n+2}-(n+1)p^{n+1})}{p-1})\right)}$$

where n and r are integers with $n \geq 2$.

Note that formulas obtained in Examples 3.1, 3.2 and 3.3 are straightforward to compute even when n and r are large. For any finite abelian p-group, an explicit formula for computing $R(G)$ was obtained in [4, Theorem 1.1]. By expanding and simplifying the formula obtained in [4, Theorem 1.1] one can compare the computations for $R(C_p^{(r)} \times C_{p^2} \times C_{p^n})$; the recursive formula used in Example 3.3 provides a more efficient method for obtaining $R(C_p^{(r)} \times C_{p^2} \times C_{p^n})$.

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